Let $\mathbb{R}_n(n \geq 2)$ be an $n$-dimensional Euclidean space, and let $S$ be any set of points in $\mathbb{R}_n$. There exist a number of instances in which the following question has an interesting answer. Suppose a property $A$ holds on each $(n-1)$-dimensional linear section $S_{n-1}$ of $S$. What additional property $B$ assumed to hold on each section $S_{n-1}$ will ensure that property $A$ holds on $S$?

The following terminology is used. A continuum is a compact connected set which may include the degenerate case of a single point. Also compactness includes closure. A generalized continuum is a set which is connected and closed. An $(n-r)$-dimensional linear section of a set $S$ with an $(n-r)$-dimensional Euclidean hyperplane $L_{n-r}$ is defined to be the set $S \cdot L_{n-r}$. A subscript will always designate the dimensionality of the set.

1. Theorems on closed, open and bounded sets. The following theorem illustrates the theory, and plays an important role in a succeeding theorem. It is a case in which condition $B$ is sufficient but not necessary. We shall always assume $n \geq 2$.

**Theorem 1.** Let $S$ be any set in $\mathbb{R}_n(n \geq 2)$. If each $(n-1)$-dimensional linear section of $S$ is connected and closed, then $S$ is closed.

**Proof.** Suppose $S$ is not closed. Then there exists a point $p \notin S$ which is a limit point of $S$. Let $L_{n-1}$ be an $(n-1)$-dimensional hyperplane containing $p$, such that $S \cdot L_{n-1} \neq 0$. Since, by hypothesis, $S_{n-1} = S \cdot L_{n-1}$ is closed, there exists an $(n-1)$-dimensional closed cube $C_{n-1} \subseteq L_{n-1}$, which contains $p$ in its interior, and for which $C_{n-1} \cdot S_{n-1} = 0$. Let $P_n$ be an $n$-dimensional hyperprism passing through $C_{n-1}$, and perpendicular to $L_{n-1}$. Since $p$ is a limit point of $S$ which is not in $S$, and since $S_{n-1}$ is closed, there exists a sequence of points $p_i \in S \cdot P_n$, such that $p_i \in L_{n-1}$, and such that $p_i \rightarrow p$ as $i \rightarrow \infty$. Let $L_{n-2}$ be any $(n-2)$-dimensional hyperplane contained in $L_{n-1}$ such that $S \cdot L_{n-2} \neq 0$, and such that $L_{n-2} \cdot C_{n-1} = 0$. Then there exists a sequence of hyperplanes $L_{n-2}$ determined by $L_{n-2}$ and $p_i$. By hypothesis each set $S \cdot L_{n-2}$ is connected. Hence since $p_i \in S \cdot L_{n-2} \cdot P_n$, and since any point $q \in S \cdot L_{n-2} \cdot L_{n-1}$ is not in $P_n$, the connectedness of $S \cdot L_{n-1}$ im-

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plies that $p^i$ and $q$ can be joined by a connected subset of $S \cdot L_{n-1}^i$ which intersects the boundary of the prism, $B(P_n)$. Hence let $r^i \subseteq S \cdot L_{n-1}^i \cdot B(P_n)$. Since the set $\{r^i\}$ is infinite, and since the prism $P_n$ has a finite number of $(n-1)$-dimensional plane faces, there exist an infinite subset of $\{r^i\}$, namely $\{r_i^i\}$, which lie on one face of $P_n$. Designate this face by $F_{n-1}^\ast$, so that $r_i^i \subseteq F_{n-1}^\ast \cdot L_{n-1}^i$. Furthermore since $p^i \to p$ as $i \to \infty$, the set $\{r_i^i\}$ lies on a bounded portion of $F_{n-1}^\ast$.

Hence since $L_{n-1}^i \to L_{n-1}$ as $i \to \infty$, the set $\{r_i^i\}$ has a limit point $r$ existing in $L_{n-1}$. Since $r_i^i \subseteq F_{n-1}^\ast \cdot L_{n-1}^i$, and since by hypothesis $S \cdot L_{n-1}^i$ is closed, $r \in S \cdot F_{n-1}^\ast$. Hence $r \in S$. But $r \in F_{n-1}^\ast \cdot L_{n-1} \subseteq C_{n-1}$, which is a contradiction, since by construction $C_{n-1} \cdot S_{n-1} = 0$. Thus the indirect proof is completed, and Theorem 1 is proved.

**COROLLARY 1.1.** If each two-dimensional plane section of $S$ is connected and closed, then $S$ is closed.

**COROLLARY 1.2.** If each $(n-1)$-dimensional linear section of $S$ is a generalized continuum, then $S$ is a generalized continuum.

In Corollary 1.2, the connectedness of $S$ is well known [6, p. 64].

This second corollary is an illustration where no additional hypotheses $B$ are needed on linear sections in order to guarantee property $A$ on $S$.

**THEOREM 2.** Let $S$ be any set in $\mathbb{R}_n (n \geq 2)$. Suppose that relative to each $(n-1)$-dimensional hyperplane $L_{n-1}$, the set $S \cdot L_{n-1}$ is an open one with a connected complement. Then $S$ is open.

**PROOF.** Let $S_{n-1}^i$ be any linear section determined by $L_{n-1}^i$. Since $S_{n-1}^i$ is open in $L_{n-1}^i$, the complement $C(S_{n-1}^i)$ is closed in $L_{n-1}^i$. Since each linear set $C(S_{n-1}^i)$ is then connected and closed, Theorem 1 implies that $C(S)$ is closed. Hence $S$ is open.

**COROLLARY 2.1.** Let $S$ be any set in $\mathbb{R}_n (n \geq 2)$. If relative to each two-dimensional plane $L_2^i$, the set $S \cdot L_2^i$ is an open one with a connected complement, then $S$ is open.

The following theorem is one in which boundedness is the principal property to be established. Here again connectedness plays an important role.

**THEOREM 3.** Let $S$ be any set in $\mathbb{R}_n (n \geq 2)$. If each $(n-1)$-dimensional linear section of $S$ is bounded and connected, then $S$ is bounded and connected.
PROOF. Choose \( p \in S \), and let \( L_{n-2} \) be a hyperplane containing the point \( p \). Consider the family of hyperplanes \( L_{n-1}^i \) passing through \( L_{n-2} \). Let \( S_n^m \) \( (m=1, 2, \cdots) \) be a set of spheres with centers at \( p \) of radii \( m \).

Suppose that \( S \) is unbounded. Then there exists a sequence of points \( x^i \in S \) \( (i=1, 2, \cdots) \) such that the distance \( \delta(p, x^i) \to \infty \) as \( i \to \infty \). Let \( L_{n-1}^i \) designate the member of \( L_{n-1}^x \) for which \( x^i \in L_{n-1}^i \). Since \( p \notin S \cdot L_{n-1}^i \cdot S_n^m \), and since, for any fixed value of \( m \), \( x^i \in S \cdot L_{n-1}^i - S_n^m \) (for sufficiently large values of \( i \)), the connectedness of \( S \cdot L_{n-1}^i \) implies that \( p \) and \( x^i \) can be joined by a connected subset of \( S \cdot L_{n-1}^i \) which intersects the boundary of \( S_n^m \), \( B(S_n^m) \). Choose \( y^m \in S \cdot L_{n-1}^i \cdot B(S_n^m) \). Since \( B(S_n^m) \) is compact, there exists a convergent subsequence \( \{y^m_i\} \) which converges to a point \( y^m \in S_n^m \), such that for the corresponding points \( x^i \), \( \delta(p, x^i) \to \infty \) as \( i \to \infty \). Without loss of generality designate \( L_{n-1}^m \) to be the member of \( L_{n-1}^x \) such that \( y^m \in L_{n-1}^m \). There exists an integer \( N \) such that when \( i > N \), \( x^i \in S \cdot L_{n-1}^i - S_{n}^{m+1} \), and such that \( \delta(p, x^i) \to \infty \) as \( i \to \infty \). Hence by the connectedness of \( S \cdot L_{n-1}^i \), there exist points \( y^{m+1, i} \in S \cdot L_{n-1}^i \cdot B(S_{n}^{m+1}) \). Since \( y^{m+1, i} \to y^m \), and since \( L_{n-1}^i \to L_{n-1}^m \) as \( i \to \infty \), there exists a convergent subsequence of \( \{y^{m+1, i}\} \) which converges to a point \( y^{m+1} \in S \cdot L_{n-1}^m \cdot S_n^{m+1} \). Since the radius of \( S_n^m \) is \( m \), by induction it follows there exists a sequence \( y^m \in S \cdot L_{n-1}^0 \), such that \( \delta(p, y^m) \to \infty \) as \( m \to \infty \).

If \( S \subseteq L_{n-1}^0 \), Theorem 3 is obviously true. Hence, suppose \( q \in S \) \( - L_{n-1}^0 \). Since \( L_{n-1}^0 \) divides \( R_n \) into two half-spaces, namely \( R_n^p \) and \( R_n^q \), suppose without loss of generality that \( q \in R_n^p \). Choose a hyperplane \( L_{n-1}^p \subset R_n^p \) so that \( L_{n-1}^p \) is parallel to \( L_{n-1}^0 \), and such that \( q \) is not on \( L_{n-1}^p \) or between \( L_{n-1}^p \) and \( L_{n-1}^0 \). Since \( y^m \in S \cdot L_{n-1}^0 \), there exists in any neighborhood of \( y^m \) a point \( \rho^m \in S \) such that \( \delta(p, \rho^m) \to \infty \) as \( m \to \infty \). Hence \( L_{n-1}^p \) and \( q \) are on opposite sides of \( L_{n-1}^p \). Join \( p \) and \( q \) by a line \( L_1^p \). Hence \( L_{n-1}^p \cap L_{n-1}^0 = \emptyset \) exists. Let \( L_{n-2}^p \subset L_{n-1}^p \) be a hyperplane such that \( r^m \in L_{n-2}^p \), and such that \( L_{n-2}^p \) is perpendicular to \( L_1^p \). The line \( L_1^p \) and the subspace \( L_{n-2}^p \) determine a hyperplane \( L_{n-1}^p \). Since \( S \cdot L_{n-1}^p \) is connected, and since \( q \) and \( \rho^m \) lie on opposite sides of \( L_{n-2}^p \) \( \in L_{n-2}^p \), we have \( S \cdot L_{n-2}^p \neq \emptyset \). Let \( s^m \in S \cdot L_{n-2}^p \). Since \( L_{n-2}^p \subset L_{n-1}^p \), then \( s^m \in L_{n-1}^p \). Since by construction \( \delta(q, y^m) \to \infty \) as \( m \to \infty \), \( \rho^m \) can be chosen so that \( \delta(q, \rho^m) \to \infty \) as \( m \to \infty \). Since \( \rho^m \to \infty \) the line \( L_1^p \) approaches parallelism to \( L_{n-1}^p \), \( \delta(q, \rho^m) \to \infty \) as \( \rho^m \to \infty \). Since \( L_{n-2}^p \) is perpendicular to \( L_1^p \), \( \delta(q, s^m) \geq \delta(q, r^m) \). Hence we have \( \delta(q, s^m) \to \infty \) as \( m \to \infty \). Since \( s^m \in L_{n-2}^p \subset L_{n-1}^p \) for all \( m \), the set \( S \cdot L_{n-1}^p \) is unbounded. This is a contradiction of hypothesis. Thus \( S \)
is bounded. Since the connectedness of \( S \) is well known, Theorem 3 is proved.

2. A characterization of star-like sets. Aumann [1] has characterized compact convex sets by means of properties on linear sections. Also Liberman [4] has made another characterization by placing properties on the set itself and also on its supporting planes. The following theorem, while restricted to two-dimensional sections, yields, as far as it goes, a generalization of Aumann’s result, for convexity is replaced by the weaker concept of star-likeness, and boundedness of the set is removed. Note that in Theorem 5 no hypotheses are placed on the set \( S \) itself. The following definition is a standard one. Refer to Brunn [2].

**Definition.** A set \( S \) is star-like with respect to a point \( a \in S \) if each straight line through \( a \) intersects \( S \) in a connected set.

In order to characterize star-like sets by linear sections the following definition of simply-connectedness in the plane is especially useful.

**Definition.** A connected plane set \( U \) is simply connected if each component of the complement of \( U \) is unbounded.

**Theorem 4.** A closed set \( S \) in \( \mathbb{R}^n \) \((n \geq 3)\) is star-like with respect to a point \( a \in S \) if and only if the following conditions hold.

1. Each two-dimensional linear section of \( S \) through the point \( a \) is a simply connected, generalized continuum.
2. For each point \( q \in S \), there exists a constant \( M > 0 \), such that each two-dimensional linear section containing \( a \) and \( q \) contains a continuum joining \( a \) and \( q \) of diameter less than \( M \).

The necessity is immediate. In particular for condition (2) note that \( M \) can be any number greater than the distance \( \delta(a,q) \).

**Sufficiency Proof.** Suppose \( S \) is not a star with respect to the point \( a \). Then since \( S \) is closed, there exist distinct points \( b \in S \), \( c \in S \), such that \( \delta(a,c) = \delta(a,b) + \delta(b,c) \), and such that the open line segment \( L_1 \) between \( b \) and \( c \) is not in \( S \). Consider any three-dimensional hyperplane \( L_3 \) such that \( L_1 \subset L_3 \). Choose a coordinate system \((x,y,z)\) in \( L_3 \) so that \( L_1 \times L_3 \) is contained in the \( x \)-axis. Let \( L_2^+ \subset L_3 \) be an open half-plane with the \( x \)-axis as an axis, whose directed normal makes a directed angle \( \theta \) with the positive \( z \)-axis. Also suppose that \( 0 \leq \theta \leq \pi \). Let \( L_2^\theta \) be the plane containing \( L_2^\theta \), and define \( L_2^\theta - L_1 \).

Designate the component of the complement of \( S_2^\theta \) that contains \( L_1 \) by \( C_2^\theta \). Since \( S_2^\theta \) is a generalized continuum, the boundary of \( C_2^\theta \) is a connected set [6, p. 117]. By a theorem in the plane [5, p. 203; 6, p. 108], the set \( C_2^\theta - L_1 \) is the sum of two mutually exclusive
open connected sets $D_2^{+}$ and $D_2^{-}$, and $L_1$ is a subset of the boundary of each of these sets. The set $D_2^{+}$ corresponds to $L_2^{+}$ in the sense that for any point $r \in L_1$, there exists a circle $R_2 \subset L_2^{+}$ with center at $r$ such that $D_2^{+} \cdot R_2 \subset L_2^{+}$ and $D_2^{-} \cdot R_2 \subset L_2^{-}$. Hypotheses (1) and (2) imply that one and only one of the sets $D_2^{+}$ and $D_2^{-}$ is unbounded. Furthermore, the bounded set, say $D_2^{+}$, is of diameter less than $M$. This is due to the fact that $D_2^{+} \subset Q$, where $Q$ is a set enclosed by the closed line segment $(a, c)$ and by the subcontinuum in $S_2^{+}$ of diameter less than $M$ which joins $a$ and $c$. Clearly $Q$ is of diameter less than $M$, whence $D_2^{+}$ is of diameter less than $M$, when it is bounded.

Remark. The set of angles $\{\alpha\}$ for which $D_2^{+}$ is bounded is closed. To prove this let $L_2^{\alpha_i} \to L_2^{\alpha}$ as $\alpha_i \to \alpha$, and suppose $D_2^{\alpha_i}$ are bounded and that $D_2^{\alpha}$ is unbounded. Choose points $r \in L_1$, and $s \in D_2^{\alpha}$ such that the distance $\delta(r, s) > M$. Since $D_2^{\alpha}$ is arcwise connected, let $A \subset D_2^{\alpha}$ be a simple arc joining $r$ and $s$, so that $A \cdot S_2^{\alpha} = 0$. Rotate $A$ rigidly in $L_2$ about $L_1$ so that $A \subset L_2^{\alpha}$ is a congruent image of $A$. By virtue of the preceding paragraph, $D_2^{\alpha_i}$ are all of diameter less than $M$. Since $A \cdot S_2^{\alpha} \neq 0$, since $A \cdot S_2^{\alpha}$ are uniformly bounded, and since $S$ is closed, we have $A \cdot S_2^{\alpha} \neq 0$. This is a contradiction; hence the remark holds. In exactly the same way, the set of angles $\{\beta\}$ for which $D_2^{\beta}$ is bounded is closed. Since the two closed sets $\{\alpha\}$ and $\{\beta\}$ cover the continuum $0 \leq \theta \leq \pi$, they have a value in common. Hence there exists a plane $L_2^{\phi}$, $0 \leq \phi \leq \pi$, such that each $D_2^{\phi}$ is bounded. But in this case $C_2^{\phi}$ would be bounded, and $S_2^{\phi}$ would not be simply connected. Hence Theorem 4 is proved.

**Corollary 4.1.** Let $S$ be a compact set in $\mathbb{R}^n$ $(n \geq 3)$. The set $S$ is a star with respect to a point $a \in S$ if and only if condition (1) in Theorem 4 holds.

Compactness of $S$ and condition (1) imply condition (2). Hence Corollary 4.1 follows from Theorem 4.

**Theorem 5.** Let $S$ be any set in $\mathbb{R}^n$ $(n \geq 3)$. The set $S$ is a closed convex set if and only if conditions (1) and (2) in Theorem 4 hold for all points $a \in S$.

The necessity is obvious. To prove the sufficiency note that Theorem 1 implies that $S$ is closed. Hence by Theorem 4, $S$ is star-like with respect to all points of $S$. Thus by definition $S$ is convex.

3. **A theorem in linear spaces.** The results of Theorem 3 can be generalized to hold in a normed, linear, metric space $\mathcal{M}$. A hyperplane
L in $\mathcal{M}$ is defined to be the set $\{x\}$ which satisfies an equation $f(x) = c$, where $f(x)$ is a linear functional, and where $c$ is a real constant. A linear section of $S$ with $L$ is the set $S \cdot L$.

**Theorem 6.** Let $S$ be any set in a normed linear metric space $\mathcal{M}$. If each linear section of $S$ is bounded and connected, then $S$ is bounded and connected.

**Proof.** Consider two independent linear functionals $f_1(x)$ and $f_2(x)$ defined on $\mathcal{M}$. Let $T$ be a transformation of the type

$$T: \quad \xi_1 = f_1(x), \quad \xi_2 = f_2(x).$$

This transformation maps $S$ in $\mathcal{M}$ into a set $S_2$ in the plane $\mathbb{R}_2$. Any linear section $S_2 \cdot L_1$ determined by the line $L_1, \alpha \xi_1 + \beta \xi_2 = \gamma$ corresponds by $T$ to the section $S \cdot L$ where $L$ is defined by $\alpha f_1(x) + \beta f_2(x) = \gamma$. Since $T$ is linear (additive and continuous), and since by hypothesis $S \cdot L$ is connected and bounded, it follows that the linear section $S_2 \cdot L_1$ is connected and bounded. Hence by Theorem 3 with $n = 2$, the set $S_2$ is bounded. Thus each functional $f_1(x)$ and $f_2(x)$ is bounded for all $x$ in $S$. Since $f_1(x)$ was an arbitrary linear functional, independent of $f_2(x)$, we have shown that all linear functionals defined on $\mathcal{M}$ are bounded on $S$. Hence by a classical theorem of uniform boundedness [3], the set $S$ is bounded. Since the connectedness is well known, Theorem 6 has been established. It should be noted that in light of Theorem 6 the proof for Theorem 3 need only have been given for $n = 2$; however, since the proof for $n$ dimensions was not appreciably longer, an elementary proof independent of the abstract boundedness theorem seemed desirable.

4. **Concluding remarks.** It should be observed that in Theorems 1–3 one cannot delete connectedness entirely, for then the theorems in general are no longer true. Theorem 5 has a preferred form since no hypotheses are placed on $S$ itself. Theorem 4 needs to be formulated so as to hold for $(n-r)$-dimensional sections. This problem is still unsolved. It should be noticed in dealing with non-compact sets that the complement of an unbounded convex set or of an unbounded star need not be connected. Hence conditions on the complement necessary to yield a characterization take on a different form than those given by Aumann [1]. The author wishes to express his gratitude to his colleagues, Professor R. H. Sorgenfrey, Professor W. T. Puckett, and Professor M. Zorn who have made helpful suggestions.

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THE SPACE $L^\omega$ AND CONVEX TOPOLOGICAL RINGS

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1. Introduction. The motive for investigating the class $L^\omega$ of functions belonging to all $L^p$-classes has no measure-theoretic origin: it was our desire to discover whether or not in every convex metric ring $R$ one could find a system $\{U\}$ of convex neighborhoods of 0 having the property that $f, g \in U$ implies $fg \in U$. We show here that $L^\omega$ has no proper convex open set $U$ containing 0 and satisfying the relation $UU \subseteq U$, thus supplying the desired counter-example.

The significance of neighborhood systems of the type $\{U\}$ described above is made somewhat clearer by a proof that they insure the existence and continuity of entire functions (for example, the exponential function) on the topological ring $R$.

Such neighborhood systems $\{U\}$ are always present in rings of continuous real-valued functions over any space, provided that convergence means uniform convergence on compact sets.

We also consider the relation of $L^\omega, L^\omega$, and the $L^p$-classes, since $L^\omega$ does not seem ever to have been discussed as a topological and algebraic entity.

2. Notation and elementary facts. Let us consider measurable functions defined on $[0, 1]$. For $p \geq 1$ we shall consistently employ the usual notation.

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More precisely, metrizable, convex, complete topological linear algebra. For these one requires continuity in both ring operations and scalar multiplication. It will appear that $L^\omega$ has these properties.