SET PROPERTIES DETERMINED BY CONDITIONS ON LINEAR SECTIONS

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Let $\mathcal{R}_n (n \geqq 2)$ be an n-dimensional Euclidean space, and let S be any set of points in $\mathcal{R}_n$. There exist a number of instances in which the following question has an interesting answer. Suppose a property A holds on each $(n-1)$-dimensional linear section $S_{n-1}{}^i$ of S. What additional property B assumed to hold on each section $S_{n-1}{}^i$ will insure that property A holds on S?

The following terminology is used. A continuum is a compact connected set which may include the degenerate case of a single point. Also compactness includes closure. A generalized continuum is a set which is connected and closed. An $(n-r)$-dimensional *linear section* of a set S with an $(n-r)$-dimensional Euclidean hyperplane L_{n-r} is defined to be the set $S \cdot L_{n-r}$. A subscript will *always* designate the dimensionality of the set.

1. Theorems on closed, open and bounded sets.

The following theorem illustrates the theory, and plays an important role in a succeeding theorem. It is a case in which condition B is sufficient but not necessary. We shall always assume $n \geqq 2$.

THEOREM 1. *Let S be any set in $\mathcal{R}_n$ $(n \geqq 2)$. If each $(n-1)$-dimensional linear section of S is connected and closed, then S is closed.*

PROOF. Suppose S is *not* closed. Then there exists a point $p \notin S$ which is a limit point of S. Let L_{n-1} be an $(n-1)$-dimensional hyperplane containing p, such that $S \cdot L_{n-1} \neq 0$. Since, by hypothesis, $S_{n-1} \equiv S \cdot L_{n-1}$ is closed, there exists an $(n-1)$-dimensional closed cube $C_{n-1} \subset L_{n-1}$, which contains p in its interior, and for which $C_{n-1} \cdot S_{n-1} = 0$. Let P_n be an n-dimensional hyperprism passing through C_{n-1}, and perpendicular to L_{n-1}. Since p is a limit point of S which is not in S, and since S_{n-1} is closed, there exists a sequence of points $p^i \in S \cdot P_n$, such that $p^i \notin L_{n-1}$, and such that $p^i \rightarrow p$ as $i \rightarrow \infty$. Let L_{n-2} be any $(n-2)$-dimensional hyperplane contained in L_{n-1} such that $S \cdot L_{n-2} \neq 0$, and such that $L_{n-2} \cdot C_{n-1} = 0$. Then there exists a sequence of hyperplanes $L_{n-1}{}^i$ determined by L_{n-2} and p^i. By hypothesis each set $S \cdot L_{n-1}{}^i$ is connected. Hence since $p^i \in S \cdot L_{n-1}{}^i \cdot P_n$, and since any point $q \in S \cdot L_{n-2} \cdot L_{n-1}{}^i$ is not in P_n, the connectedness of $S \cdot L_{n-1}{}^i$ im-

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plies that p^i and q can be joined by a connected subset of $S \cdot L_{n-1}^i$ which intersects the boundary of the prism, $B(P_n)$. Hence let $r^i \in S \cdot L_{n-1}^i \cdot B(P_n)$. Since the set $\{r^i\}$ is infinite, and since the prism P_n has a finite number of $(n-1)$-dimensional plane faces, there exist an infinite subset of $\{r^i\}$, namely $\{r^{i_j}\}$, which lie on *one* face of P_n. Designate this face by F_{n-1}^*, so that $r^{i_j} \in F_{n-1}^* \cdot L_{n-1}^{i_j}$. Furthermore since $p^{i_j} \rightarrow p$, as $i_j \rightarrow \infty$, the set $\{r^{i_j}\}$ lies on a bounded portion of F_{n-1}^*. Hence since $L_{n-1}^{i_j} \rightarrow L_{n-1}$ as $i_j \rightarrow \infty$, the set $\{r^{i_j}\}$ has a limit point r existing in L_{n-1}. Since $r^{i_j} \in F_{n-1}^* \cdot S$, and since by hypothesis $S \cdot L_{n-1}^*$ is closed, $r \in S \cdot F_{n-1}^*$. Hence $r \in S$. But $r \in F_{n-1}^* \cdot L_{n-1} \subset C_{n-1}$, which is a contradiction, since by construction $C_{n-1} \cdot S_{n-1} = 0$. Thus the indirect proof is completed, and Theorem 1 is proved.

COROLLARY 1.1. *If each two-dimensional plane section of S is connected and closed, then S is closed.*

COROLLARY 1.2. *If each $(n-1)$-dimensional linear section of S is a generalized continuum, then S is a generalized continuum.*

In Corollary 1.2, the connectedness of S is well known [6, p. 64].[1] This second corollary is an illustration where no additional hypotheses B are needed on linear sections in order to guarantee property A on S.

THEOREM 2. *Let S be any set in $\mathcal{R}_n$ $(n \geqq 2)$. Suppose that relative to each $(n-1)$-dimensional hyperplane L_{n-1}, the set $S \cdot L_{n-1}$ is an open one with a connected complement. Then S is open.*

PROOF. Let S_{n-1}^i be any linear section determined by L_{n-1}^i. Since S_{n-1}^i is open in L_{n-1}^i, the complement $C(S_{n-1}^i)$ is closed in L_{n-1}^i. Since each linear set $C(S_{n-1}^i)$ is then connected and closed, Theorem 1 implies that $C(S)$ is closed. Hence S is open.

COROLLARY 2.1. *Let S be any set in $\mathcal{R}_n$ $(n \geqq 2)$. If relative to each two-dimensional plane L_2^i, the set $S \cdot L_2^i$ is an open one with a connected complement, then S is open.*

The following theorem is one in which boundedness is the principal property to be established. Here again connectedness plays an important role.

THEOREM 3. *Let S be any set in $\mathcal{R}_n$ $(n \geqq 2)$. If each $(n-1)$-dimensional linear section of S is bounded and connected, then S is bounded and connected.*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

PROOF. Choose $p \in S$, and let L_{n-2} be a hyperplane containing the point p. Consider the family of hyperplanes L_{n-1}^{α} passing through L_{n-2}. Let $\mathfrak{S}_n^m$ $(m=1, 2, \cdots)$ be a set of spheres with centers at p of radii m.

Suppose that S is unbounded. Then there exists a sequence of points $x^i \in S$ $(i=1, 2, \cdots)$ such that the distance $\delta(p, x^i) \to \infty$ as $i \to \infty$. Let L_{n-1}^i designate the member of L_{n-1}^{α} for which $x^i \in L_{n-1}^i$. Since $p \in S \cdot L_{n-1}^i \cdot \mathfrak{S}_n^m$, and since, for any *fixed* value of m, $x^i \in S \cdot L_{n-1}^i - \mathfrak{S}_n^m$ (for sufficiently large values of i), the connectedness of $S \cdot L_{n-1}^i$ implies that p and x^i can be joined by a connected subset of $S \cdot L_{n-1}^i$ which intersects the boundary of $\mathfrak{S}_n^m$, $B(\mathfrak{S}_n^m)$. Choose $y^{m,i} \in S \cdot L_{n-1}^i \cdot B(\mathfrak{S}_n^m)$. Since $B(\mathfrak{S}_n^m)$ is compact, there exists a convergent subsequence $\{y^{m,i_j}\}$ which converges to a point $y^m \in \mathfrak{S}_n^m$, such that for the corresponding points x^{i_j}, $\delta(p, x^{i_j}) \to \infty$ as $i_j \to \infty$. Without loss of generality designate L_{n-1}^0 to be the member of L_{n-1}^{α} such that $y^m \in L_{n-1}^0$. There exists an integer N such that when $i_j > N$, $x^{i_j} \in S \cdot L_{n-1}^{i_j} - \mathfrak{S}_n^{m+1}$, and such that $\delta(p, x^{i_j}) \to \infty$ as $i_j \to \infty$. Hence by the connectedness of $S \cdot L_{n-1}^{i_j}$, there exist points $y^{m+1,i_j} \in S \cdot L_{n-1}^{i_j} \cdot B(\mathfrak{S}_n^{m+1})$. Since $y^{m,i_j} \to y^m$, and since $L_{n-1}^{i_j} \to L_{n-1}^0$ as $i_j \to \infty$, there exists a convergent subsequence of $\{y^{m+1,i_j}\}$ which converges to a point $y^{m+1} \in \bar{S} \cdot L_{n-1}^0 \cdot \mathfrak{S}_n^{m+1}$. Since the radius of $\mathfrak{S}_n^m$ is m, by induction it follows there exists a sequence $y^m \in \bar{S} \cdot L_{n-1}^0$, such that $\delta(p, y^m) \to \infty$ as $m \to \infty$.

If $S \subset L_{n-1}^0$, Theorem 3 is obviously true. Hence, suppose $q \in S - L_{n-1}^0$. Since L_{n-1}^0 divides $\mathfrak{R}_n$ into two half-spaces, namely $\mathfrak{R}_n^+$ and $\mathfrak{R}_n^-$, suppose without loss of generality that $q \in \mathfrak{R}_n^+$. Choose a hyperplane $L_{n-1}^+ \subset \mathfrak{R}_n^+$ so that L_{n-1}^+ is *parallel* to L_{n-1}^0, and *such that q is not on L_{n-1}^+ or between L_{n-1}^+ and L_{n-1}^0*. Since $y^m \in \bar{S} \cdot L_{n-1}^0$ there exists in any neighborhood of y^m a point $p^m \in S$, such that p^m and q are on *opposite sides* of L_{n-1}^+. Join p^m and q by a line L_1^*. Hence $L_1^* \cdot L_{n-1}^+ \equiv r^m$ exists. Let $L_{n-2}^* \subset L_{n-1}^+$ be a hyperplane such that $r^m \in L_{n-2}^*$, and such that L_{n-2}^* is perpendicular to L_1^*. The line L_1^* and the subspace L_{n-2}^* determine a hyperplane L_{n-1}^*. Since $S \cdot L_{n-1}^*$ is connected, and since q and p^m lie on opposite sides of L_{n-2}^* in L_{n-1}^*, we have $S \cdot L_{n-2}^* \neq 0$. Let $s^m \in S \cdot L_{n-2}^*$. Since $L_{n-2}^* \subset L_{n-1}^+$, then $s^m \in L_{n-1}^+$. Since by construction $\delta(q, y^m) \to \infty$ as $m \to \infty$, p^m can be chosen so that $\delta(q, p^m) \to \infty$ as $m \to \infty$. Since as $p^m \to \infty$ the line L_1^* approaches parallelism to L_{n-1}^+, $\delta(q, r^m) \to \infty$ as $p^m \to \infty$. Since L_{n-2}^* is perpendicular to L_1^*, $\delta(q, s^m) \geqq \delta(q, r^m)$. Hence we have $\delta(q, s^m) \to \infty$ as $m \to \infty$. Since $s^m \in L_{n-2}^* \subset L_{n-1}^+$ for all m, the set $S \cdot L_{n-1}^+$ is *unbounded*. This is a contradiction of hypothesis. Thus S

is bounded. Since the connectedness of S is well known, Theorem 3 is proved.

2. **A characterization of star-like sets.** Aumann [1] has characterized compact convex sets by means of properties on linear sections. Also Liberman [4] has made another characterization by placing properties on the set itself and also on its supporting planes. The following theorem, while restricted to two-dimensional sections, yields, as far as it goes, a generalization of Aumann's result, for convexity is replaced by the weaker concept of star-likeness, and boundedness of the set is removed. Note that in Theorem 5 *no* hypotheses are placed on the set S itself. The following definition is a standard one. Refer to Brunn [2].

DEFINITION. *A set S is star-like with respect to a point $a \in S$ if each straight line through a intersects S in a connected set.*

In order to characterize star-like sets by linear sections the following definition of simply-connectedness in the plane is especially useful.

DEFINITION. *A connected plane set U is simply connected if each component of the complement of U is unbounded.*

THEOREM 4. *A closed set S in $\mathcal{R}_n$ $(n \geqq 3)$ is star-like with respect to a point $a \in S$ if and only if the following conditions hold.*

(1) *Each two-dimensional linear section of S through the point a is a simply connected, generalized continuum.*

(2) *For each point $q \in S$, there exists a constant $M > 0$, such that each two-dimensional linear section containing a and q contains a continuum joining a and q of diameter less than M.*

The necessity is immediate. In particular for condition (2) note that M can be any number greater than the distance $\delta(a, q)$.

SUFFICIENCY PROOF. Suppose S is *not* a star with respect to the point a. Then since S is closed, there exist distinct points $b \in S$, $c \in S$, such that $\delta(a, c) = \delta(a, b) + \delta(b, c)$, and such that the open line segment L_1 between b and c is not in S. Consider any three-dimensional hyperplane L_3 such that $L_1 \subset L_3$. Choose a coordinate system (x, y, z) in L_3 so that L_1 is contained in the x-axis. Let $L_2^{\theta +} \subset L_3$ be an open half-plane with the x-axis as an axis, whose directed normal makes a directed angle θ with the positive z-axis. Also suppose that $0 \leqq \theta \leqq \pi$. Let L_2^θ be the plane containing $L_2^{\theta +}$, and define $L_2^{\theta -} \equiv L_2^\theta - \overline{L}_2^{\theta +}$.

Designate the component of the complement of $S_2^\theta \equiv S \cdot L_2^\theta$ which contains L_1 by C_2^θ. Since S_2^θ is a generalized continuum, the boundary of C_2^θ is a connected set [6, p. 117]. By a theorem in the plane [5, p. 203; 6, p. 108], the set $C_2^\theta - L_1$ is the sum of two mutually exclusive open connected sets $D_2^{\theta+}$ and $D_2^{\theta-}$, and L_1 is a subset of the boundary of each of these sets. The set $D_2^{\theta+}$ corresponds to $L_2^{\theta+}$ in the sense that for any point $r \in L_1$, there exists a circle $R_2 \subset L_2^{\theta}$ with center at r such that $D_2^{\theta+} \cdot R_2 \subset L_2^{\theta+}$ and $D_2^{\theta-} \cdot R_2 \subset L_2^{\theta-}$.

Hypotheses (1) and (2) imply that *one and only one* of the sets $D_2^{\theta+}$ and $D_2^{\theta-}$ is unbounded. Furthermore, the bounded set, say $D_2^{\theta+}$, is of diameter less than M. This is due to the fact that $D_2^{\theta+} \subset Q$, where Q is a set enclosed by the closed line segment (a, c) and by the subcontinuum in S_2^{θ} of diameter less than M which joins a and c. Clearly Q is of diameter less than M, whence $D_2^{\theta+}$ is of diameter less than M, when it is bounded.

Remark. The set of angles $\{\alpha\}$ for which $D_2^{\alpha+}$ is bounded is closed.

To prove this let $L_2^{\alpha_i} \to L_2^{\alpha}$ as $\alpha_i \to \alpha$, and suppose $D_2^{\alpha_i+}$ are bounded and that $D_2^{\alpha+}$ is unbounded. Choose points $r \in L_1$, and $s \in D_2^{\alpha+}$ such that the distance $\delta(r, s) > M$. Since $D_2^{\alpha+}$ is arcwise connected, let $A \subset D_2^{\alpha+}$ be a simple arc joining r and s, so that $A \cdot S_2^{\alpha} = 0$. Rotate A rigidly in L_3 about L_1 so that $A^{\alpha_i} \subset L_2^{\alpha_i}$ is a congruent image of A. By virtue of the preceding paragraph, $D_2^{\alpha_i+}$ are all of diameter less than M. Since $A^{\alpha_i} \cdot S_2^{\alpha_i} \neq 0$, since $A^{\alpha_i} \cdot S_2^{\alpha_i}$ are uniformly bounded, and since S is closed, we have $A \cdot S_2^{\alpha} \neq 0$. This is a contradiction; hence the remark holds. In exactly the same way, the set of angles $\{\beta\}$ for which $D_2^{\beta-}$ is bounded is closed. Since the two closed sets $\{\alpha\}$ and $\{\beta\}$ cover the continuum $0 \leqq \theta \leqq \pi$, they have a value in common. Hence there exists a plane L_2^{ϕ}, $0 \leqq \phi \leqq \pi$, such that each $D_2^{\phi-}$ and $D_2^{\phi+}$ is bounded. But in this case C_2^{ϕ} would be bounded, and S_2^{ϕ} would not be simply connected. Hence Theorem 4 is proved.

COROLLARY 4.1. *Let S be a compact set in $\mathcal{R}_n$ $(n \geqq 3)$. The set S is a star with respect to a point $a \in S$ if and only if condition* (1) *in Theorem* 4 *holds.*

Compactness of S and condition (1) imply condition (2). Hence Corollary 4.1 follows from Theorem 4.

THEOREM 5. *Let S be any set in $\mathcal{R}_n$ $(n \geqq 3)$. The set S is a closed convex set if and only if conditions* (1) *and* (2) *in Theorem* 4 *hold for all points $a \in S$.*

The necessity is obvious. To prove the sufficiency note that Theorem 1 implies that S is closed. Hence by Theorem 4, S is star-like with respect to *all* points of S. Thus by definition S is convex.

3. **A theorem in linear spaces.** The results of Theorem 3 can be generalized to hold in a normed, linear, metric space $\mathfrak{M}$. A hyperplane

L in $\mathfrak{M}$ is defined to be the set $\{x\}$ which satisfies an equation $f(x)=c$, where $f(x)$ is a linear functional, and where c is a real constant. A linear section of S with L is the set $S\cdot L$.

THEOREM 6. *Let S be any set in a normed linear metric space $\mathfrak{M}$. If each linear section of S is bounded and connected, then S is bounded and connected.*

PROOF. Consider two independent linear functionals $f_1(x)$ and $f_2(x)$ defined on $\mathfrak{M}$. Let T be a transformation of the type

$$T: \qquad \xi_1 = f_1(x), \qquad \xi_2 = f_2(x).$$

This transformation maps S in $\mathfrak{M}$ into a set S_2 in the plane $\mathcal{R}_2$. *Any* linear section $S_2\cdot L_1$ determined by the line $L_1, \alpha\xi_1+\beta\xi_2=\gamma$ corresponds by T to the section $S\cdot L$ where L is defined by $\alpha f_1(x)+\beta f_2(x)=\gamma$. Since T is linear (additive and continuous), and since by hypothesis $S\cdot L$ is connected and bounded, it follows that the linear section $S_2\cdot L_1$ is connected and bounded. Hence by Theorem 3 with $n=2$, the set S_2 is bounded. Thus each functional $f_1(x)$ and $f_2(x)$ is bounded for *all* x in S. Since $f_1(x)$ was an *arbitrary* linear functional, independent of $f_2(x)$, we have shown that *all* linear functionals defined on $\mathfrak{M}$ are bounded on S. Hence by a classical theorem of uniform boundedness [3], the set S is bounded. Since the connectedness is well known, Theorem 6 has been established. It should be noted that in light of Theorem 6 the proof for Theorem 3 need only have been given for $n=2$; however, since the proof for n dimensions was not appreciably longer, an elementary proof independent of the abstract boundedness theorem seemed desirable.

4. **Concluding remarks.** It should be observed that in Theorems 1–3 one cannot delete connectedness entirely, for then the theorems in general are no longer true. Theorem 5 has a preferred form since no hypotheses are placed on S itself. Theorem 4 needs to be formulated so as to hold for $(n-r)$-dimensional sections. This problem is still unsolved. It should be noticed in dealing with non-compact sets that the complement of an unbounded convex set or of an unbounded star need *not* be connected. Hence conditions on the complement *necessary* to yield a characterization take on a different form than those given by Aumann [1]. The author wishes to express his gratitude to his colleagues, Professor R. H. Sorgenfrey, Professor W. T. Puckett, and Professor M. Zorn who have made helpful suggestions.

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THE SPACE L^ω AND CONVEX TOPOLOGICAL RINGS

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1. **Introduction.** The motive for investigating the class L^ω of functions belonging to all L^p-classes has no measure-theoretic origin: it was our desire to discover whether or not in every convex metric ring[1] R one could find a system $\{U\}$ of convex neighborhoods of 0 having the property that $f, g \in U$ implies $fg \in U$. We show here that L^ω has no proper convex open set U containing 0 and satisfying the relation $UU \subset U$, thus supplying the desired counter-example.

The significance of neighborhood systems of the type $\{U\}$ described above is made somewhat clearer by a proof that they insure the existence and continuity of entire functions (for example, the exponential function) on the topological ring R.

Such neighborhood systems $\{U\}$ are always present in rings of continuous real-valued functions over any space, provided that convergence means uniform convergence on compact sets.

We also consider the relation of L^∞, L^ω, and the L^p-classes, since L^ω does not seem ever to have been discussed as a topological and algebraic entity.

2. **Notation and elementary facts.** Let us consider measurable functions defined on $[0, 1]$. For $p \geqq 1$ we shall consistently employ the usual notation

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[1] More precisely, metrizable, convex, complete topological linear algebra. For these one requires continuity in both ring operations and scalar multiplication. It will appear that L^ω has these properties.