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THE SPACE L^ω AND CONVEX TOPOLOGICAL RINGS

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1. **Introduction.** The motive for investigating the class L^ω of functions belonging to all L^p-classes has no measure-theoretic origin: it was our desire to discover whether or not in every convex metric ring[1] R one could find a system $\{U\}$ of convex neighborhoods of 0 having the property that $f, g \in U$ implies $fg \in U$. We show here that L^ω has no proper convex open set U containing 0 and satisfying the relation $UU \subset U$, thus supplying the desired counter-example.

The significance of neighborhood systems of the type $\{U\}$ described above is made somewhat clearer by a proof that they insure the existence and continuity of entire functions (for example, the exponential function) on the topological ring R.

Such neighborhood systems $\{U\}$ are always present in rings of continuous real-valued functions over any space, provided that convergence means uniform convergence on compact sets.

We also consider the relation of L^∞, L^ω, and the L^p-classes, since L^ω does not seem ever to have been discussed as a topological and algebraic entity.

2. **Notation and elementary facts.** Let us consider measurable functions defined on $[0, 1]$. For $p \geqq 1$ we shall consistently employ the usual notation

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[1] More precisely, metrizable, convex, complete topological linear algebra. For these one requires continuity in both ring operations and scalar multiplication. It will appear that L^ω has these properties.

$$\|f\|_p = \left(\int_0^1 |f(x)|^p dx\right)^{1/p}$$

even when the right side is infinite.

Therefore L^p consists of all functions f for which $\|f\|_p$ is less than ∞.

L^ω evidently consists of all functions f for which $\|f\|_1, \|f\|_2, \cdots, \|f\|_p, \cdots$ are all finite.

Because of the relation[2]

$$\text{(H)} \qquad \|fg\|_p \leqq \|f\|_q \cdot \|g\|_r, \qquad 1/p = 1/q + 1/r,$$

one has

$$\|f\|_1 \leqq \|f\|_2 \leqq \cdots,$$

since the measure of $[0, 1]$ is 1. Therefore we may take the sets of functions f,

$$\|f\|_p < e$$

where $p \geqq 1$ and $e > 0$, as neighborhoods of 0 in L^ω. These neighborhoods are convex because

$$\|\lambda f + \mu g\|_p \leqq \lambda\|f\|_p + \mu\|g\|_p < e$$

when $\lambda, \mu \geqq 0, \lambda + \mu = 1$, and $\|f\|_p, \|g\|_p < e$. Therefore addition is continuous in L^ω and, by relation (H), multiplication is also.

Multiplication is not generally possible in L^p.

Now the inequalities above imply that the limit

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$$

always exists. (It may be infinite.) Those f's for which $\|f\|_\infty$ is finite form a set usually called L^∞, and $\|f\|_\infty$ is taken as a norm in L^∞. We shall employ the known fact that $\|f\|_\infty$ is also the least number h such that $|f(x)| > h$ only on a set of measure zero.

Multiplication in L^∞ is continuous, since

$$\|fg\|_\infty \leqq \|f\|_\infty \|g\|_\infty,$$

from which it follows that if U is any sphere about 0, contained in the unit sphere of L^∞, then $UU \subset U$.

3. **The relation of L^∞, L^ω, and L^p.** These spaces are related by successive proper inclusion.

THEOREM 1. *$L^\infty \subset L^\omega \subset L^p$ but $L^\infty \neq L^\omega \neq L^p$. The identity mappings*

[2] Cf. E. J. McShane, *Integration*, Princeton, 1944, for most of the facts which we assume. A formula equivalent to (H) appears on p. 186.

$L^\infty \to L^\omega \to L^p$ are continuous, but their inverses are not. L^∞ is dense in L^ω, and L^ω is dense in each L^p.

PROOF. The inclusions and the continuity of the mappings are obvious.

If we define $l(x) = |\log x|$, then l does not belong to L^∞. Since $\|l\|_p = (p!)^{1/p}$, $l \in L^p$ for each $p \geqq 1$, and hence $l \in L^\omega$. Thus $L^\omega \neq L^\infty$.

Similarly, the function with values $x^{-1/2p}$ belongs to L^p, but not to L^{2p}, and hence not to L^ω.

Now let $l_n(x) = n^{-1}|\log x|$ or n, whichever is the smaller. Then $\|l_n - 0\|_p < n^{-1}\|l\|_p$ which tends to zero as $n \to \infty$; but $\|l_n - 0\|_\infty = n$, $n \to \infty$. Thus the inverse of the mapping $L^\infty \to L^\omega$ is not continuous.

A similar process applied to the function $x^{-1/2p}$ yields a sequence which converges to zero in L^p but not in L^{4p}, and thus not in L^ω.

Finally, suppose $f \in L^\omega$ be given. Define

$$f_n(x) = \begin{cases} -n & \text{when } f(x) < -n, \\ f(x) & \text{when } -n \leqq f(x) \leqq n, \\ n & \text{when } n < f(x). \end{cases}$$

Then $f_n \to f$ in each L^p and hence in L^ω. Since the f_n are taken from L^∞ the latter is dense in L^ω and in each L^p, which establishes the third sentence of the theorem.

L^ω can be metrized, so as to be complete, by

$$(f, g) = \sum_{p=1}^{\infty} \frac{2^{-p}\|f - g\|_p}{1 + \|f - g\|_p}.$$

4. **Multiplication in L^ω.** By relation (H), this is continuous. The following theorem shows the divergence between its properties and those of normed rings.

THEOREM 2. *L^ω is a convex metric commutative ring with the property that if U is a convex open set in L^ω containing* 0, *and if $UU \subset U$, then U coincides with the whole space L^ω.*

PROOF. There exists a $p \geqq 1$ and an $e > 0$ such that $\|f\|_p \leqq e$ implies $f \in U$. Therefore a function f having values not greater than h on a set of measure not greater than $(e/h)^p$, and vanishing elsewhere, must lie in U, together with all its powers $f^2, f^3, \cdots$.

Let $h = 2$, and set $m = (e/2)^p$, for brevity.

Consider any function g which has the value b on a set S of measure a, and vanishes elsewhere. Suppose k is any integer such that $a \leqq mk$. Select an integer n such that $bk \leqq 2^n$. Now we can cover S by k nonoverlapping subsets of measure not greater than m and define functions $f_1, \cdots, f_k$, where f_i has the value $(bk)^{1/n}$ on the ith subset of S, and vanishes elsewhere. Thus $f_1, \cdots, f_k \in U$, and also $f_1^n, \cdots, f_k^n \in U$. Since U is convex

$$g = \frac{1}{k} f_1^n + \cdots + \frac{1}{k} f_k^n$$

must belong to U.

Now any function g' assuming only a finite number of values is a linear combination, with positive constants *whose sum is* 1, of such functions as g. Therefore these functions lie in U.

Since these functions g' are known to be dense in L^∞ and thus in L^ω, we have U a dense, open convex set in L. Thus $U = L^\omega$.

COROLLARY. *The topology assigned to L^ω cannot be defined by any norm.*

5. **Entire functions in rings.** Of course Theorem 2 shows more about L^ω than is needed for a counter-example to the proposition mentioned in the introduction, as will appear from the following theorem, and the fact that $e^{|\log x|} = 1/x$ is not summable, while $|\log x|$, as we have seen, lies in L^ω.

THEOREM 3. *If R is a complete topological ring with a complete system $\{U\}$ of convex neighborhoods of zero each satisfying $UU \subset U$, and*

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

is a power series representing an entire function, then, for each $f \in R$,

$$P(f) = a_0 + a_1 f + a_2 f^2 + \cdots$$

converges, and P is a continuous operation on R into itself.

In particular, for the exponential function, if U is convex, contains zero, and $UU \subset U$, then

$$e^U \subset 1 + 2U.$$

PROOF. Let us first show that $P(f)$ converges. Therefore, suppose U is any neighborhood of the system $\{U\}$. Let $f \in R$.

Then for some $t > 0$, $tf \in U$. Hence $(tf)^2, (tf)^3, \cdots$ will all lie in U.

Further, let us find m_0 so large that for $m \geqq m_0$

$$|a_m t^{-m}| + |a_{m+1} t^{-m-1}| + \cdots$$

is less than 1. Then, since U is convex, we can deduce that for $n > m > m_0$,

$$a_m t^{-m}(tf)^m + \cdots + a_n t^{-n}(tf)^n$$

or its equivalent

$$a_m f^m + \cdots + a_n f^n$$

must lie in U.

Since R is assumed complete, $P(f)$ converges to a limit.

The continuity of P can be proved as follows:

$$D = P(f + h) - P(f) = \sum_{n=0}^{n} a_{n+1} g_{n+1}$$

where

$$g_n = (f + h)^{n+1} - f^{n+1}.$$

Let U be a neighborhood of the system $\{U\}$, and suppose $f/t \in U$ where $0 < t < \infty$. Select a real number a,

$$a > |a_1| (t + 1) + |a_2| (t + 1)^2 + \cdots, \qquad a \geqq 1,$$

and require h to be so close to zero that $ah \in U$.

There is no point in writing down the expansion of g_n since terms cannot be collected when R is not commutative. However, each term will contain h, and if g_n is written as a sum of products of powers of f/t and h, the coefficients will add up to $(t+1)^n - t^n$.

Since f/t and ah lie in U, and $UU \subset U$, we have

$$h_n = (t + 1)^{-n} a g_n \in U,$$

where, before dividing, we have replaced $(t+1)^n - t^n$ by $(t+1)^n$. Now D is a linear combination of $h_1, h_2, \cdots$ with coefficients whose absolute values add up to less than 1, and since U is convex we conclude $D \in U$.

Therefore P is continuous at f.

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