ON THE LOWER ORDER OF INTEGRAL FUNCTIONS

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Let $f(z)=\sum_0^\infty a_n z^n$ be an integral function of order ρ. It is known that[1]

$$(1)\quad \limsup_{n\to\infty} \frac{n \log n}{\log \{1/|a_n|\}} = \rho = \limsup_{r\to\infty} \frac{\log \log M(r)}{\log r} \quad (0 \leqq \rho \leqq \infty).$$

A similar result for the lower[2] order λ, namely

$$\liminf_{n\to\infty} \frac{n \log n}{\log \{1/|a_n|\}} = \lambda = \liminf_{r\to\infty} \frac{\log \log M(r)}{\log r},$$

does not always hold. In fact for

$$\exp(z^2) + \exp(z) = 2 + z + z^2\left(\frac{1}{1!} + \frac{1}{2!}\right) + \cdots,$$

$$\liminf_{n\to\infty} \frac{n \log n}{\log \{1/|a_n|\}} = 1$$

whereas $\lambda=\rho=2$.

We prove here the following theorem.

THEOREM 1. *If $f(z)=\sum_0^\infty a_n z^n$ is an integral function of order ρ and lower order λ $(0\leqq\lambda\leqq\infty)$ then*

$$(2)\qquad \lambda \geqq \liminf_{n\to\infty} \frac{n \log n}{\log \{1/|a_n|\}} \geqq \liminf_{n\to\infty} \frac{\log n}{\log |a_n/a_{n+1}|}.$$

COROLLARY 1.[3]

$$(3)\quad \begin{aligned} &\liminf_{n\to\infty} \frac{\log |a_n/a_{n+1}|}{\log n} \leqq \liminf_{n\to\infty} \frac{\log \{1/|a_n|\}}{n \log n} = \frac{1}{\rho} \leqq \frac{1}{\lambda} \\ &\leqq \limsup_{n\to\infty} \frac{\log \{1/|a_n|\}}{n \log n}; \leqq \limsup_{n\to\infty} \frac{\log |a_n/a_{n+1}|}{\log n}. \end{aligned}$$

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[1] E. C. Titchmarsh, *Theory of functions*, pp. 253–254; E. T. Copson, *Theory of functions of a complex variable*, pp. 175–178.

[2] For the definition, and so on, see (i) J. M. Whittaker, *The lower order of integral functions*, J. London Math. Soc. vol. 8 (1933) pp. 20–27; (ii) S. M. Shah, *The lower order of the zeros of an integral function* (II), Proceedings of the Indian Academy of Sciences (A) vol. 21 (1945) pp. 162–164.

[3] Cf. a similar result (1) in S. M. Shah, *The maximum term of an entire series*, Mathematics Student vol. 10 (1942) pp. 80–82.

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COROLLARY 2. *If* $\lim_{n\to\infty} n \log n/\log\{1/|a_n|\} = L$ *where* $0 \leqq L < \infty$ *then* $f(z) = \sum_0^\infty a_n z^n$ *is an integral function of regular growth*[4] *and of order* L.

THEOREM 2. *If* (i) $f(z) = \sum_0^\infty a_n z^n$ *is an integral function of order* ρ *and lower order* λ $(0 \leqq \lambda \leqq \infty)$ *such that* (ii) $|a_n/a_{n+1}|$ *is a nondecreasing function of* n *for* $n > n_0$, *then*

$$\lambda = \liminf_{n\to\infty} \frac{n \log n}{\log \{1/|a_n|\}} = \liminf_{n\to\infty} \frac{\log n}{\log |a_n/a_{n+1}|}, \tag{4}$$

$$\rho = \limsup_{n\to\infty} \frac{\log n}{\log |a_n/a_{n+1}|}. \tag{5}$$

We note that the hypothesis (ii) of Theorem 2 does not imply that $f(z)$ is of regular growth. In fact we have the following theorem.

THEOREM 3. *There exists an integral function* $f(z) = \sum_0^\infty a_n z^n$ *for which* (i) $a_n > 0$, (ii) a_n/a_{n+1} *is a steadily increasing function of* n, *and* (iii) $\rho > \lambda$.

An interesting application of these results can be made to the series $F(z) = \sum_0^\infty a_n \epsilon_n z^n$ where $\{\epsilon_n\}$ are a set of numbers such that $|\epsilon_n| = 1$ or 0 and such that $\sum_0^\infty a_n \epsilon_n z^n$ consists of an infinite number of terms. $F(z)$ is an integral function. Let its order be $\rho(F)$ and lower order be $\lambda(F)$. Since

$$M(r, f) \geqq |a_n| r^n \geqq |a_n \epsilon_n| r^n$$

for every n and r, and so if $\mu(r)$ denotes the maximum term, $M(r, f) \geqq \mu(r, F)$. Hence

$$\lambda(f) \geqq \lambda(F); \qquad \rho(f) \geqq \rho(F). \tag{6}$$

If $|a_n/a_{n+1}| = \psi(n)$ (say) is a nondecreasing function of n then

$$\lambda(f) = \liminf_{n\to\infty} \frac{n \log n}{\log \{1/|a_n|\}} \leqq \limsup_{n\to\infty} \frac{n \log n}{\log \{1/|a_n \epsilon_n|\}} = \rho(F) \tag{7}$$

and so we have the following theorem.

THEOREM 4. *If* $f(z) = \sum_0^\infty a_n z^n$ *is an integral function of order* ρ *and of lower order* λ *and is such that* $|a_n/a_{n+1}|$ *is a nondecreasing function of* n *for* $n > n_0$, *then* $F(z) = \sum_0^\infty a_n \epsilon_n z^n$ *is of order* $\rho(F) \geqq \lambda$.

For instance every function $F = \sum_0^\infty \epsilon_n z^n/n!$ is of order 1.

An example, to illustrate the point that by an appropriate choice

[4] Cf. G. Valiron, *Lectures on the general theory of integral functions*, pp. 41–44.

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of ϵ_n the order $\rho(F)$ of $F(z)=\sum a_n\epsilon_n z^n$ can be made equal to any number x where $\lambda(f)\leqq x\leqq\rho(f)$, is given in the proof of Theorem 3.

The function $\exp z=\sum_0^\infty z^n/n!$ for which $\psi(n)$ is an increasing function of n is bounded on the real negative axis and the series

$$F(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

is bounded on the real axis. *If $\psi(n)$ is increasing sufficiently rapidly then we prove that $f(z)$ and $F(z)$ are not bounded on any line* $\arg z=\alpha$ $(0\leqq\alpha\leqq 2\pi)$. In fact we have the following theorem.

THEOREM 5. *If $f(z)=\sum_0^\infty a_n z^n$ is an integral function of lower order λ such that $|a_n/a_{n+1}|\geqq\vartheta^2|a_{n-1}/a_n|$ for $n>n_0$ then*

$$(8)\quad \limsup_{r\to\infty}\frac{\log\log m(r,f)}{\log r}\geqq\lambda;\qquad \limsup_{r\to\infty}\frac{\log\log m(r,F)}{\log r}\geqq\lambda$$

where $m(r,f)=\min_{|z|=r}|f(z)|$ and $\vartheta=2\cdot 2$.

LEMMA. *a_n is any sequence of real or complex numbers such that*[5]

$$\text{(i)}\qquad |a_n| < 1 \qquad \textit{for } n > n_0.$$

Let

$$\theta(n) = \frac{\log\{1/|a_n|\}}{n\log n};\qquad \phi(n) = \frac{\log|a_n/a_{n+1}|}{\log n};$$

$$\alpha = \liminf_{n\to\infty}\phi(n);\qquad \gamma = \liminf_{n\to\infty}\{1/\phi(n)\};$$

$$\beta = \limsup_{n\to\infty}\phi(n);\qquad \delta = \limsup_{n\to\infty}\{1/\phi(n)\};$$

$$A = \liminf_{n\to\infty}\theta(n);\qquad C = \liminf_{n\to\infty}\{1/\theta(n)\};$$

$$B = \limsup_{n\to\infty}\theta(n);\qquad D = \limsup_{n\to\infty}\{1/\theta(n)\};$$

then

$$(9)\qquad \alpha\leqq A = 1/D;\qquad 1/C = B\leqq\beta;\qquad C\geqq\gamma.$$

(ii) *If*[5] *further $\psi(n)$ is a nondecreasing function of n for $n\geqq N$ and $\psi(N)\geqq 1$ then*

$$(10)\qquad C=\gamma=1/\beta;\qquad D=\delta=1/\alpha.$$

The proof of (9) is straightforward and omitted.

[5] Some of the relations in (9) and (10) hold under less restrictive conditions.

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PROOF OF (10). By hypothesis (ii), α, β, γ and δ are non-negative and $\beta=1/\gamma$, $\alpha=1/\delta$. We prove $B\geqq\beta$. Suppose first $0<\beta<\infty$. Then

$$\psi(n) > n^{\beta-\epsilon} \qquad \text{for } n = N_1, N_2, \cdots, N_p, \cdots.$$

Let $N_1>\max\{n_0, N\}$. Then

$$\left|\frac{1}{a_n}\right| = k(N_1)\psi(N_1+1)\cdots\psi(n-1),$$

$$\theta(n) = o(1) + \frac{\log\psi(N_1+1)+\cdots+\log\psi(n-1)}{n\log n}.$$

Let $n=[N_p\log^2 N_p]+1$. Then

$$\theta(n) \geqq o(1) + \frac{(n-N_p)\log N_p^{\beta-\epsilon}}{n\log n}.$$

Hence $B\geqq\beta$ which holds also when $\beta=0$. If β be infinite the above argument with an arbitrary large number instead of $\beta-\epsilon$ gives that $B=\infty$. Hence from (9) we get that $B=\beta$ and so $C=\gamma=1/\beta$. The second relation in (10) follows similarly.

PROOF OF THEOREM 1. Since $\sum a_n$ is convergent, $|a_n|<1$ for $n>n_0$. As $C\geqq\gamma$ we need prove $\lambda\geqq C$ only. Suppose first $0<C<\infty$. Then

$$\left.\begin{aligned}\frac{n\log n}{\log\{1/|a_n|\}} &> C-\epsilon,\\ |a_n| &> n^{-n/(C-\epsilon)},\end{aligned}\right\} \qquad \text{for all } n \geqq N(\epsilon).$$

Let $r_n=2n^{1/(C-\epsilon)}$. If $r_n\leqq r\leqq r_{n+1}$ $(n>N)$ then

$$M(r) \geqq |a_n|r^n \geqq |a_n|r_n^n > n^{-n/(C-\epsilon)}\exp(n\log r_n) = \exp(n\log 2).$$

Hence $\log M(r)\geqq\log 2\{(r/2)^{C-\epsilon}-1\}$ for all large r and so $\lambda\geqq C$, which holds when $C=0$. If $C=\infty$, the above argument shows that $\lambda=\infty$.

Corollary 1 follows from (1), (2) and (9), and Corollary 2 from (1) and (2). The example given at the beginning of the paper shows that $f(z)$ may be of regular growth and $\lim_{n\to\infty}\{n\log n/\log\{1/|a_n|\}\}$ may not exist.

PROOF OF THEOREM 2. Let $\mu(r)$ denote the maximum term, $\nu(r)$ its rank. By hypothesis (ii), $\psi(n)>\psi(n-1)$ for an infinity of n; for if otherwise $\psi(n)=\psi(n+1)=\cdots$ ad inf for $n>p$, say, and hence the radius of convergence of the series $\sum a_n z^n$ would be finite. $\psi(n)$ tends to infinity with n.

When $\psi(n)>\psi(n-1)$ the term $a_n z^n$ becomes a maximum term

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and we have $\mu(r)=|a_n|r^n$, $\nu(r)=n$ for $\psi(n-1)\leqq r<\psi(n)$. Now $\lambda=\lim\inf_{r\to\infty}\log\nu(r)/\log r$. Suppose first that $0<\lambda<\infty$. Then $\nu(r)>r^{\lambda-\epsilon}$ for $r>R=R(\epsilon)$. Let $|z|=r>R$ and let $a_{m_1}z^{m_1}$ and $a_{m_2}z^{m_2}$ $(m_1>n_0;\ \psi(m_1-1)>R)$ be two consecutive terms so that $m_1\leqq m_2-1$ and let $m_1<n\leqq m_2$. Since $a_{m_1}z^{m_1}$ is maximum term we have $\nu(r)=m_1$ for $\psi(m_1-1)\leqq r<\psi(m_1)$. Hence for every r in this interval $m_1=\nu(r)>r^{\lambda-\epsilon}$. In particular $m_1>\{\psi(m_1)-C\}^{\lambda-\epsilon}$ where $C=\min\{1, ((\psi(m_1)-\psi(m_1-1))/2\}$. Further we have

$$\psi(m_1) = \psi(1+m_1) = \cdots = \psi(n-1).$$

Hence

$$\psi(n_0+1)\cdots\psi(n-1) = \left|\frac{a_{n_0+1}}{a_n}\right| \leqq \{\psi(n-1)\}^{n-n_0-1}$$
$$< \{C + m_1^{1/(\lambda-\epsilon)}\}^{n-n_0-1}$$
$$< K(n_0)2^n n^{(n-n_0-1)/(\lambda-\epsilon)}.$$

Hence for all large n

$$\left|\frac{1}{a_n}\right| < K_1(n_0)2^n\cdot n^{(n-n_0-1)/(\lambda-\epsilon)}$$

and so

$$C \geqq \lambda \tag{11}$$

which holds when $\lambda=0$. If $\lambda=\infty$ the above argument gives $C=\infty$. Hence from (2), $\lambda=C$ and so from (10) we get (4); and from (1) and (10) we have (5).

PROOF OF THEOREM 3. Let $n_1=2$, $n_{s+1}=n_s^4$ $(s=1, 2, 3, \cdots)$,

$$r_1 = 1, \qquad r_m = m \qquad \text{for } n_s \leqq m < n_s^2,$$

$$r_m = n_{s+1} - \frac{n_{s+1}-m}{\{(n_{s+1})!\}^{(n_{s+1})!}} \qquad \text{for } n_s^2 \leqq m < n_{s+1},$$

$s=1, 2, 3, \cdots$, and let

$$f(z) = 1 + \sum_1^\infty \frac{z^n}{r_1r_2\cdots r_n}.$$

Then $a_n>0$ and $a_n/a_{n+1}=r_{n+1}$ which is a steadily increasing function of n. Also

$$\theta(n) = \frac{\log r_1 + \cdots + \log r_n}{n\log n}.$$

Hence

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$$\theta(n_{s+1}) \sim \frac{(n_s^4 - n_s^2)\log(n_s^4)}{4n_s^4 \log n_s} \sim 1,$$

$$\theta([n_s^2 \log n_s]) \sim \frac{(n_s^2 \log n_s - n_s^2)\log(n_s^4) + O(n_s^2 \log n_s)}{n_s^2 \log n_s^4 \log \{n_s^2 \log n_s\}} \sim 2.$$

It is easily seen that $\lim \sup_{n\to\infty}\theta(n) = 2$; $\lim \inf_{n\to\infty}\theta(n) = 1$. Hence $f(z)$ is an integral function of order 1 and lower order 1/2. Let now

$$\epsilon_m = \begin{cases} 1 & \text{when} \quad m = [n_s^2 \log n_s] \qquad (s = 1, 2, 3, \cdots) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$F(z) = \sum_1^\infty a_n \epsilon_n z^n = \sum_1^\infty \frac{\epsilon_n z^n}{r_1 r_2 \cdots r_n}$$

is an integral function of order 1/2. If

$$\epsilon_m = \begin{cases} 1 & \text{when} \quad m = n_s \qquad (s = 1, 2, 3, \cdots) \\ 0 & \text{otherwise} \end{cases}$$

then $F(z)$ is of order 1. Let $1/2 < x < 1$ and $\epsilon_m = 1$ when $m = [\exp(4x \log n_s)]$ $(s = 1, 2, 3, \cdots)$ and zero otherwise; then $F(z)$ is of order x.

PROOF OF THEOREM 5. Let $|\epsilon_n| = 1$ for $n = N_1, N_2, \cdots N_p, \cdots$ $(N_1 > n_0)$. We write $N_p = N$. Let $R_N = \vartheta\psi(N-1)$ and $|z| = R_N = R$.

$$\mu(r, f) = |a_N| r^N = \mu(r, F) \qquad \text{for } \psi(N-1) \leqq r < \psi(N)$$

and R lies inside this interval.

$$\begin{aligned} |f(z)| &= \left| \sum_0^{N-1} a_n z^n + a_N z^N + \sum_{N+1}^\infty a_n z^n \right| \\ &\geqq \mu(R, f) - \left| \sum_0^{N-1} a_n z^n \right| - \left| \sum_{N+1}^\infty a_n z^n \right|. \end{aligned}$$

Now

$$\begin{aligned} \left| \sum_0^{N-1} a_n z^n \right| &\leqq |a_{N-1}| R^{N-1} + \cdots \\ &\leqq \mu(R) \left\{ \frac{1}{\vartheta} + \frac{1}{\vartheta^4} + \frac{1}{\vartheta^9} + \cdots + \frac{1}{\vartheta^{(N-n_0-2)^2}} + o(1) \right\} \\ &\leqq \mu(R) \left\{ \frac{1}{\vartheta} + \frac{1}{\vartheta^4} + \frac{1}{\vartheta^9} + \cdots \text{ ad inf} \right\} + \frac{\mu(R)}{10^{10}} \end{aligned}$$

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for all large N.

$$\left|\sum_{N+1}^{\infty} a_n z^n\right| \leqq |a_{N+1}| R^{N+1} + \cdots$$
$$\leqq \mu(R)\left\{\frac{1}{\theta} + \frac{1}{\theta^4} + \frac{1}{\theta^9} + \cdots\right\}.$$

Hence for all large R

$$|f(z)| > \frac{\mu(R, f)}{10000}.$$

Similarly

$$|F(z)| > \frac{\mu(R, f)}{10000}.$$

Hence f and F are not bounded on any line $\arg z = \alpha$.

Since

$$\liminf_{r\to\infty} \frac{\log \log \mu(r, f)}{\log r} = \lambda$$

the theorem follows.

Added in proof. A short note containing a part of each of the Theorems 1, 2, and 3 appeared in J. Indian Math. Soc. vol. 9 (1945) pp. 50–54.

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