A PROPERTY OF DERIVATIVES

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If a real function defined over a closed interval \([a, b]\) is differentiable at each point of the interval, it is well known that its derivative possesses the Darboux property: if \(f'(c) < \xi < f'(d)\), then there is a point \(e\) between \(c\) and \(d\) with \(f'(e) = \xi\).

Now let \(\alpha, \beta\), with \(\alpha < \beta\), be any two fixed reals, and consider the set \(E(\alpha, \beta) = E\{x/\alpha < f'(x) < \beta\}\). It is easily seen, as a consequence of the Darboux property, that any such set \(E(\alpha, \beta)\) must contain a continuum of points, unless it is empty. The question of the measure of \(E(\alpha, \beta)\) does not seem to be covered in the literature, except in the case in which the given interval is either \((-\infty, \beta)\) or \((\alpha, +\infty)\). We prove that any such set \(E(\alpha, \beta)\) is either empty or of positive measure.

We remark that this result cannot be deduced from the Darboux property alone; Lebesgue exhibited a function\(^1\) which possesses that property without satisfying the measure condition. Another example is the following. Let \(C\) be the Cantor closed nondense set of measure zero and power \(c\) in the unit interval, and let \(\{T_n\} (n = 1, 2, 3, \ldots)\) be a sequence of linear transformations such that the sets \(T_n(C)\) are disjoint, and such that any sub-interval of \([0, 1]\) contains some \(T_n(C)\). We take \(T_1\) to be the identity. Let the function \(g(x)\) be defined on \(C\) in such a way as to assume all values from zero to one inclusive; on \(T_n(C)\) let \(g(x) = g(T_n^{-1}(x))\). On all remaining points of the unit interval set \(g(x) = 0\). It is clear that this function \(g\) possesses the Darboux property, but that the set \(E\{x/1/2 < g(x) < 1\}\) will be nonvoid and of measure zero.

**Theorem.** If \(f(x)\) is real and everywhere differentiable in the closed interval \([a, b]\), then for any two reals \(\alpha, \beta\) \((\alpha < \beta)\), the set

\[E(\alpha, \beta) = E\{x/\alpha < f'(x) < \beta\}\]

is empty or of positive measure.

**Proof.** We start with the following known result:\(^2\) if a continuous function \(f(x)\) is differentiable in the interval \([a, b]\), with the possible exception of a denumerable set of points \(x\), and if \(f'(x)\) is non-negative almost everywhere, then \(f(x)\) is nondecreasing. It follows that if \(f'(x)\) exists for all \(x\) in \([a, b]\), and \(f'(x) \geq \lambda\) [or \(f'(x) \leq \mu\)] for almost all \(x\),

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then \( f'(x) \geq \lambda \) \( [f'(x) \leq \mu] \) for all \( x \) in \( [a, b] \) (by considering the functions \( f-\lambda x, f-\mu x \)). In other words, if a function is differentiable everywhere, then the least upper and greatest lower bounds of its derivative are not changed if sets of measure zero are neglected in computing them: this is the theorem which results from our formulation if either \( \alpha = -\infty \) or \( \beta = +\infty \). We shall make use of this remark in the following argument.

Consider \( \alpha, \beta \) arbitrary and finite: write \( E \) for \( E(\alpha, \beta) \). Suppose \( E \) not void, and \( mE = 0 \); we shall show that a contradiction results. We write \( E_\alpha = E \{ x/f'(x) \leq \alpha \} \), \( E_\beta = E \{ x/f'(x) \geq \beta \} \), so that the interval \( [a, b] = E + E_\alpha + E_\beta \), the three sets being disjoint.

First, \( E \subseteq E_\alpha \cdot E_\beta \). To prove this relation, assume the contrary and let \( x_0 \) belong to \( E \) but not to \( E_\alpha \); then \( x_0 \) is in the interior of some interval \( U \) void. But then in the interval \( U \) the set of points where \( f' < \beta \) is not empty, since it contains \( x_0 \), but is of measure zero, since it is a subset of \( E \). This contradicts the special case of our theorem which was referred to in our preliminary remark. Thus \( E \subseteq E_\alpha \), and by a similar argument, \( E \subseteq E_\beta \) also.

The derivative \( f'(x) \) belongs to Baire's first class, and hence, by a known theorem, if \( A \) is any subset of \( [a, b] \), and \( f' \) be considered for the moment on the domain \( A \) alone, its points of discontinuity must form a set of the first category relative to \( A \). Take for the subset \( A \) the closure of our set \( E \): this set, \( \overline{E} \), being closed, is of second category relative to itself, and hence we shall have a contradiction if we show that \( f'(x) \) is everywhere discontinuous considered on the domain \( \overline{E} \). This may be seen as follows: let \( x_0 \) be a point, first, of \( E \) itself. Since \( E \subseteq E_\alpha \cdot E_\beta \), we have for any interval \( I \) which contains \( x_0 \)

\[
\sup_{x \in I} f'(x) \geq \beta, \quad \inf_{x \in I} f'(x) \leq \alpha.
\]

Because \( f' \) possesses the Darboux property, we infer

\[
\sup_{x \in I \cdot \overline{E}} f'(x) = \beta, \quad \inf_{x \in I \cdot \overline{E}} f'(x) = \alpha,
\]

so that we have now shown that, considered on domain \( \overline{E} \), and so a fortiori on domain \( \overline{E} \), the function \( f' \) is discontinuous at each point of \( E \). On domain \( \overline{E} \) the saltus of \( f' \) at each point of \( E \) has been shown to be at least \( \beta - \alpha \); the same, then, will be true at each point of \( \overline{E} \). The function \( f' \) has now been shown to be everywhere discontinuous considered on domain \( \overline{E} \); this completes the proof.

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* Kuratowsky, Topologie, Warsaw, 1933, p. 189.