THE BEHAVIOR OF MEASURE AND MEASURABILITY UNDER CHANGE OF SCALE IN WIENER SPACE

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1. Introduction. Denoting by C the space of all real-valued continuous functions $x(t)$ on $0 \leqq t \leqq 1$ which vanish at $t=0$, we consider the magnification, or change of scale,

$$y(\cdot) = \lambda x(\cdot) \tag{1.1}$$

for λ a nonvanishing real number. An investigation of this transformation leads to some very surprising results. For example, we find that there exists a subset C_1 of C with measure equal to the measure of the total space,

$$m_w(C_1) = m_w(C), \tag{1.2}$$

which is transformed into a set of measure zero by every transformation of the form (1.1) except when $\lambda = \pm 1$. Thus,

$$m_w(\lambda C_1) = 0 \qquad \text{for } -\infty < \lambda < \infty, \lambda \neq \pm 1, \tag{1.3}$$

where λC_1 denotes the set of all functions $y = \lambda x$ for which $x \in C_1$.

This result is based upon the following theorem which we prove in §2:

THEOREM 1. *Denote by* $\sigma_n(x)$ *the sum*

$$\sigma_n(x) = \sum_{j=1}^{2^n} \left\{ x\left(\frac{j}{2^n}\right) - x\left(\frac{j-1}{2^n}\right) \right\}^2. \tag{1.4}$$

Then for almost all x *the limit*

$$\lim_{n\to\infty} \sigma_n(x) = \int_0^1 | dx(t) |^2$$

exists, and

$$\int_0^1 | dx(t) |^2 = 1/2 \qquad \text{for almost all } x. \tag{1.5}$$

Using Theorem 1 and certain lemmas to be proved in §§3 and 4, we also obtain the following theorem, which we number as Theorem 3.

THEOREM 3. *Let* $f(\lambda)$ *be a given function of* λ *defined for all positive* λ *and satisfying* $0 \leqq f(\lambda) \leqq 1$. *Then by an explicit construction (without the*

Presented to the Society, April 27, 1946; received by the editors August 6, 1946.

use of Zermelo's axiom) we can construct a specific set E such that E is measurable under all magnifications and $m_w(\lambda E) = f(\lambda)$ for all $\lambda > 0$.

We shall prove this theorem in §4.

Throughout this note we shall use the measure on C defined by Wiener.[1]

2. **Proof of Theorem** 1. Let m be any positive integer, and let

$$0 = t_0 < t_1 < \cdots < t_m < 1. \tag{2.1}$$

Later we shall take m to be of the form 2^n, and $t_j = j/2^n$; for the present no such specialization need be made. Denote by $s_m(x)$ the sum

$$s_m(x) = \sum_{j=1}^{m} \{x(t_j) - x(t_{j-1})\}^2. \tag{2.2}$$

We shall now evaluate the integral

$$\int_C^w \{s_m(x) - 1/2\}^2 d_w x = \int_C^w s_m^2(x) d_w x - \int_C^w s_m(x) d_w x + 1/4. \tag{2.3}$$

Each of the two integrals on the right can be expressed as an m-fold Lebesgue integral which in turn can be easily evaluated. Thus

$$\begin{aligned} \int_C^w s_m(x) d_w x &= \pi^{-m/2}[t_1(t_2 - t_1) \cdots (t_m - t_{m-1})]^{-1/2} \\ &\quad \cdot \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{j=1}^{m} (\xi_j - \xi_{j-1})^2 \\ &\quad \cdot \exp\left\{-\sum_{j=1}^{m} \frac{(\xi_j - \xi_{j-1})^2}{t_j - t_{j-1}}\right\} d\xi_1 \cdots d\xi_m \\ &= \pi^{-m/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{j=1}^{m} \eta_j^2 (t_j - t_{j-1}) \\ &\quad \cdot \exp\left\{-\sum_{j=1}^{m} \eta_j^2\right\} d\eta_1 \cdots d\eta_m \\ &= \frac{1}{2} \sum_{j=1}^{m} (t_j - t_{j-1}) = \frac{1}{2} \cdot \end{aligned} \tag{2.4}$$

[1] *Generalized harmonic analysis*, Acta Math. vol. 55 (1930) pp. 117–258, especially pp. 214–234. See also other references to Wiener's work given in this paper. (For further properties of Wiener measure and the Wiener integral, see *Transformations of Wiener integrals under a general class of linear transformations* by R. H. Cameron and W. T. Martin, Trans. Amer. Math. Soc. vol. 58 (1945) pp. 184–219; see also other references there.)

Similarly

$$
\begin{aligned}
\int_C^w s_m^2(x)d_w x &= \pi^{-m/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\sum_{j=1}^{m}\eta_j^2(t_j - t_{j-1})\right]^2 \\
&\quad\cdot\exp\left\{-\sum_{j=1}^{m}\eta_j^2\right\}d\eta_1\cdots d\eta_m \\
&= \pi^{-m/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\sum_{j=1}^{m}\eta_j^4(t_j - t_{j-1})^2\right. \\
&\quad\left. + 2\sum_{j=1}^{m-1}\sum_{p=j+1}^{m}\eta_j^2\eta_p^2(t_j - t_{j-1})(t_p - t_{p-1})\right] \\
&\quad\cdot\exp\left\{-\sum_{j=1}^{m}\eta_j^2\right\}d\eta_1\cdots d\eta_m \\
&= \frac{3}{4}\sum_{j=1}^{m}(t_j - t_{j-1})^2 \\
&\quad + \frac{1}{2}\sum_{j=1}^{m-1}\sum_{p=j+1}^{m}(t_j - t_{j-1})(t_p - t_{p-1}).
\end{aligned}
\tag{2.5}
$$

On specializing the t_j to be

$$t_j = j/m, \qquad j = 0, 1, \cdots, m, \tag{2.6}$$

and inserting (2.4) and (2.5) into (2.3) we obtain

$$
\begin{aligned}
&\int_C^w \left\{s_m(x) - \frac{1}{2}\right\}^2 d_w x \\
&\qquad = \frac{3}{4m} + \frac{1}{2m^2}\cdot\frac{m(m-1)}{2} - \frac{1}{2} + \frac{1}{4} = \frac{1}{2m}\cdot
\end{aligned}
\tag{2.7}
$$

This relation enables us to prove Theorem 1; letting m range over the subsequence 2^n, $n=1, 2, 3, \cdots$, we see that (2.7) yields

$$\int_C^w \left[\sigma_n(x) - \frac{1}{2}\right]^2 d_w x = 2^{-n-1}. \tag{2.8}$$

A familiar argument yields the convergence almost everywhere of $\sigma_n(x)$ to the value $1/2$. In fact, denote by E_n the set of x's for which

$$|\sigma_n(x) - 1/2| \geqq 2^{-n/3}. \tag{2.9}$$

Then by (2.8)

$$m(E_n) \leqq 2^{-n/3}. \tag{2.10}$$

Hence if $F_n = E_n + E_{n+1} + E_{n+2} + \cdots$,

$$m(F_n) \leqq 2^{-n/3} + 2^{-(n+1)/3} + \cdots < 6 \cdot 2^{-n/3}. \tag{2.11}$$

But on $C - F_n$

$$| \sigma_j(x) - 1/2 | < 2^{-j/3} \qquad \text{for } j = n, n+1, \cdots. \tag{2.12}$$

Hence for each positive integer n, we have

$$\lim_{j\to\infty} \sigma_j(x) = 1/2 \qquad \text{for } x \text{ on } C - F_n. \tag{2.13}$$

Thus

$$\lim_{j\to\infty} \sigma_j(x) = 1/2 \tag{2.14}$$

except possibly in the set $F_1F_2F_3 \cdots$. But

$$m_w(F_1F_2F_3 \cdots) \leqq m_w(F_n) < 6 \cdot 2^{-n/3}, \qquad n = 1, 2, 3, \cdots. \tag{2.15}$$

Thus $m_w(F_1F_2F_3 \cdots) = 0$ and (2.14) holds for almost all x. This concludes the proof of Theorem 1.

3. **Invariant and non-invariant null sets.** We now introduce the notion of an invariant null set.

DEFINITION. A set E is called an *invariant null set* if for all $\lambda > 0$

$$m_w(\lambda E) = 0 \tag{3.1}$$

where λE is the set consisting of all functions $y(\cdot) = \lambda x(\cdot)$ for which $x(\cdot) \in E$.

Notation. The set of $x(t)$ in C such that $\int_0^1 | dx(t) |^2$ does not exist will be denoted by D.

For each $\lambda \geqq 0$, the set of $x(t)$ in C such that

$$\int_0^1 | dx(t) |^2 = \lambda^2/2$$

will be denoted by C_λ.

LEMMA 1. *The sets C_0 and D are invariant null sets.*

PROOF. Clearly $\lambda C_0 = C_0$ and $\lambda D = D$ while by Theorem 1

$$m_w(C_0) = m_w(D) = 0.$$

LEMMA 2. *The measures of the sets C_λ are as follows:*

$$m_w(C_1) = 1, \qquad m_w(C_\lambda) = 0 \quad \text{if} \quad \lambda \neq 1, \qquad m_w\Big(\sum_{0<\lambda\neq 1} C_\lambda\Big) = 0.$$

PROOF. The first statement follows immediately from Theorem 1. Then the third statement follows from the first (since $\sum_{0<\lambda\neq 1} C_\lambda$ is contained in the complement of C_1 which is a null set) and the second follows from the third.

LEMMA 3. *The sets C_λ (for $\lambda \geqq 0$) and D are all disjoint, and their sum is C.*

This follows mmediately from the definitions of C_λ and D.

LEMMA 4. *For all non-negative λ and positive μ*

$$\mu C_\lambda = C_{\mu\lambda}.$$

PROOF. Obviously we have $\sigma_n(\mu x) = \mu^2 \sigma_n(x)$, and hence if $x \in C_\lambda$,

$$\lim_{n\to\infty} \sigma_n(\mu x) = \mu^2 \lim_{n\to\infty} \sigma_n(x)$$

so that $\int_0^1 |d(\mu x(t))|^2 = \mu^2\lambda^2/2$, and $\mu x \in C_{\mu\lambda}$.

LEMMA 5.

$$m_w\left(\frac{1}{\lambda} C_\lambda\right) = 1 \qquad \textit{for all } \lambda > 0,$$

$$m_w\left(\frac{1}{\mu} C_\lambda\right) = 0 \qquad \textit{if } \mu \neq \lambda;\ \mu > 0,\ \lambda \geqq 0;$$

$$m_w\left(\sum_{0<\mu\neq\lambda} \frac{1}{\mu} C_\lambda\right) = 0 \qquad \textit{for each fixed } \lambda \geqq 0.$$

These three statements are immediate consequence of the three corresponding statements in Lemma 2 and the transformation property of Lemma 4.

COROLLARY 1. *In the decomposition of C into disjoint sets:*

$$C = D + C_0 + \sum_{0<\lambda\neq 1} C_\lambda + C_1$$

there are only two invariant null sets, namely D and C_0. The set C_1 is of measure 1, while the other sets C_λ $(0<\lambda\neq 1)$ are non-invariant null sets.

4. **Construction of sets having prescribed measures after change of scale.** In this section we prove Theorem 3 and some related results.

LEMMA 6. *Given two real numbers λ and μ satisfying the inequalities $0 \leqq \lambda \leqq \mu \leqq 1$, there exists a subset E of C such that*

$$m_{iw}(E) = \lambda \quad \text{and} \quad m_{ew}(E) = \mu.$$

We know that there exists a subset F of the real numbers s on the interval I: $(0 \leqq s \leqq 1)$ such that $m_i(F) = \lambda$ and $m_e(F) = \mu$. Now apart from null sets, the Wiener mapping[2] T takes I into C in an essentially 1-1 measure-preserving way, and since inner measure can be defined in terms of the sup of the measure of suitable (measurable) sets and outer measure can similarly be defined in terms of the inf of the measure of suitable (measurable) sets, it follows that the set $E = TF$ in the space C has the desired property.

THEOREM 2. *Let $f(\lambda)$ and $g(\lambda)$ be two functions defined on the set of all positive λ and completely arbitrary except that*

$$0 \leqq f(\lambda) \leqq g(\lambda) \leqq 1 \qquad \textit{on } 0 < \lambda.$$

Then there exists a set E such that

$$(4.1) \qquad m_{iw}(\lambda E) = f(\lambda) \quad \textit{and} \quad m_{ew}(\lambda E) = g(\lambda) \qquad \textit{for all } \lambda > 0.$$

PROOF. By Lemma 6 and Zermelo's axiom there exists a one-parameter set of sets E_λ such that $m_{iw}(E_\lambda) = f(\lambda)$ and $m_{ew}(E_\lambda) = g(\lambda)$. Define $\tilde{E}_\lambda = C_1 E_\lambda$. Then

$$E_\lambda - \tilde{E}_\lambda = E_\lambda(C - C_1),$$

and since $C - C_1$ is a null set (cf. Lemma 2), so is $E_\lambda - \tilde{E}_\lambda$. Thus $\tilde{E}_\lambda$ differs from E_λ by a null set and hence has the same inner and outer measure as E_λ:

$$(4.2) \qquad m_{iw}(\tilde{E}_\lambda) = f(\lambda), \qquad m_{ew}(\tilde{E}_\lambda) = g(\lambda), \qquad 0 < \lambda.$$

Now define

$$(4.3) \qquad E = \sum_{0<\mu} \left(\frac{1}{\mu} \tilde{E}_\mu \right).$$

We shall show that E has the required properties, (4.1). Consider

$$\lambda E = \sum_{0<\mu} \left(\frac{\lambda}{\mu} \tilde{E}_\mu \right) = \tilde{E}_\lambda + \sum_{0<\mu\neq\lambda} \left(\frac{\lambda}{\mu} \tilde{E}_\mu \right).$$

Now since $\tilde{E}_\mu \subset C_1$, we have by Lemma 4

$$\sum_{0<\mu\neq\lambda} \left(\frac{\lambda}{\mu} \tilde{E}_\mu \right) \subset \sum_{0<\mu\neq\lambda} \left(\frac{\lambda}{\mu} C_1 \right) = \sum_{0<\mu\neq\lambda} \left(\frac{1}{\mu} C_\lambda \right),$$

and thus by Lemma 5

$$\sum_{0<\mu\neq\lambda} \left(\frac{\lambda}{\mu} \tilde{E}_\mu \right)$$

[2] Loc. cit.

is a null set. Thus λE and $\tilde{E}_\lambda$ differ by a null set and hence have the same inner and outer measures. Hence, by (4.2), (4.1) holds with E defined as in (4.3).

COROLLARY 2. *Let $f(\lambda)$ be an arbitrary function satisfying $0 \leqq f(\lambda) \leqq 1$ on $0<\lambda$. Then there is a set E which is measurable under every magnification and such that $m_w(\lambda E) = f(\lambda)$, $0<\lambda$.*

We now strengthen Corollary 2 by showing that in this case the set E can be constructed explicitly (without Zermelo's axiom). This is the content of Theorem 3 as stated in the introduction.

To prove Theorem 3, consider the quasi-interval Q_ξ with only one subdivision $t_1 = 1$ and with the corresponding lower and upper bounds ξ and ∞ respectively. Thus

$$Q_\xi: \qquad \xi < x(1).$$

Now

$$m_w(Q_\xi) = \frac{1}{\pi^{1/2}} \int_\xi^\infty e^{-v^2} dv = \pi^{-1/2} \operatorname{Erfc}(\xi).$$

It is clear that $\operatorname{Erfc}(\xi)$ is a decreasing function of ξ all the way from $-\infty$ to ∞, decreasing from $\pi^{1/2}$ at $-\infty$ to 0 at ∞. Thus $\operatorname{Erfc}^{-1}(w)$ is a single-valued decreasing function on $0 \leqq w \leqq \pi^{1/2}$, decreasing from $\operatorname{Erfc}^{-1}(0) = \infty$ to $\operatorname{Erfc}^{-1}(\pi^{1/2}) = -\infty$. Thus if v is any given real number on $0 \leqq v \leqq 1$, there is an explicitly constructed number ξ, namely

$$\xi = \operatorname{Erfc}^{-1}(\pi^{1/2} v)$$

such that the (explicitly constructed) set Q_ξ has the measure

$$m_w(Q_\xi) = v.$$

From here the proof goes as in Theorem 2: Let $E_\lambda = Q_{\xi_\lambda}$, where

$$\xi_\lambda = \operatorname{Erfc}^{-1}(\pi^{1/2} f(\lambda)).$$

Then let $\tilde{E}_\lambda = E_\lambda C_1$ and

$$E = \sum_{0<\mu} \left(\frac{1}{\mu} \tilde{E}_\mu \right).$$

It is clear that E has been explicitly constructed. Also

$$m_w(\tilde{E}_\lambda) = m_w(E_\lambda) = f(\lambda),$$

and as before

$$\lambda E = \tilde{E}_\lambda + \sum_{0<\mu\neq\lambda} \left(\frac{\lambda}{\mu} \tilde{E}_\mu \right)$$

and

$$m_w\left(\sum_{0<\mu\neq\lambda} \frac{\lambda}{\mu} \tilde{E}_\mu\right) = 0,$$

so $m_w(\lambda E) = m_w(\tilde{E}_\lambda) = f(\lambda)$, and the theorem is proved.

5. **The final theorem.** Our final theorem is the following paradoxical sounding theorem.

THEOREM 4. *Let $F_\lambda(x)$ be a given one-parameter family of functionals, there being one functional for each positive value of λ and each functional being defined on C.*

Then there exists an explicitly constructed functional $F(x)$ such that for each positive λ

$$F(\lambda x) = F_\lambda(x) \qquad \textit{almost everywhere on } C. \tag{5.1}$$

For the proof we merely let

$$F(x) = F_\lambda\left(\frac{x}{\lambda}\right) \qquad \text{on } C_\lambda \text{ for all } \lambda > 0 \tag{5.2}$$

and take

$$F(x) = 0 \text{ on } C_0 \text{ and on } D.$$

By Lemma 3, these sets are all disjoint and hence there is no x for which $F(x)$ has been defined twice (that is, the definition is self-consistent), and again by Lemma 3 the sum of these sets is C, and hence $F(x)$ is defined for every x in C. Moreover the definition is entirely explicit (and hence Zermelo's axiom has not been used explicitly or implicitly).

Now for all positive λ and μ we have by (5.2)

$$F(\mu x) = F_\lambda\left(\frac{\mu x}{\lambda}\right) \tag{5.3}$$

for $\mu x \in C_\lambda$, that is, for x on $(1/\mu)C_\lambda = (C_{\lambda/\mu})$. Taking $\mu = \lambda$, we have

$$F(\lambda x) = F_\lambda(x) \qquad \text{on } C_1 \text{ for each } \lambda > 0. \tag{5.4}$$

But $m_w(C_1) = 1$, and hence for almost all x in C we have

$$F(\lambda x) = F_\lambda(x) \qquad \text{for each } \lambda > 0.$$

Thus Theorem 4 is proved.

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