OPEN TRANSFORMATIONS AND DIMENSION

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This paper considers separable metric spaces $A$ and $B$ and open transformations. If, for each $x \in A$, $f(x) \in B$ and the image under $f$ of every open set in $A$ is a set open in $B$, then $f$ is an open transformation. Continuity of $f$ is not assumed. Such transformations have been studied by Rhoda Manning [1].

Theorem 1. If $f(A) = B$ where $f$ is open, then there exists a subset $A_1$ of $A$ such that (1) $f(A_1) = B$, (2) for $y \in B$, the set $f^{-1}(y) \cdot A_1$ is countable, and (3) $f$, considered as a transformation of $A_1$ into $B$, is open.

Proof. Let $K_1, K_2, \ldots$ denote the elements of a countable base (open sets) for the space $A$. For every $y \in B$ and each $i$ let $P_{yi}$ be a point of $K_i \cdot f^{-1}(y)$, provided this set is non vacuous. Let $A_1$ be the set of all points $P_{yi}$ so obtained. Properties (1) and (2) are obvious. To prove (3), let $V$ be an open set in $A_1$, and $U$ an open set in $A$ such that $U \cdot A_1 = V$. Now for every $y$ the set $f^{-1}(y) \cdot A_1$ is dense in $f^{-1}(y)$. Hence if $f^{-1}(y)$ has a point in $U$ then it has a point in $A_1 \cdot U$ so $f(V) = f(U)$ is an open set in $B$.

Theorem 2. There exist countable-fold open mappings which increase dimension.

Proof. There exist open mappings which increase dimension [2]. Theorem 2 follows by applying Theorem 1 to any such example.

Theorem 3. If $\dim A = n$ and $-1 < m \leq n$, then there exists a $B$ and a transformation $f$ such that (1) $f(A) = B$, (2) $f$ is open and 1-1, and (3) $\dim B = m$. In other words, dimension can be lowered at will by a 1-1 open transformation.

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1 One statement in the abstract (Bull. Amer. Math. Soc. Abstract 59-5-210) is incorrect. Theorem 4 gives the correct statement.

2 Numbers in brackets refer to the bibliography.

3 A mapping is a continuous transformation.

4 Alexandroff [5] has proved that if $A$ is compact then no countable-fold open mapping can increase dimension.

5 Compare the following special case of a theorem of Hurewicz [4, p. 91, Theorem VI 7]): "If $f$ is a closed mapping of $A$ into $B$ and for each $y \in B$, $f^{-1}(y)$ is zero-dimensional, then $\dim B \geq \dim A$." In a footnote (loc. cit.) the authors state that it is not known if Theorem VI 7 is true for open mappings. The answer is in the negative and their example VI 10 is a counter example, as the mapping $f$ is actually open.
PROOF. It is sufficient to prove the theorem for \( m = n - 1 \). For if \( f(A) = B \) and \( g(B) = C \), where \( f \) and \( g \) are open and 1-1, then the product \( gf \) is open and 1-1 and transforms \( A \) onto \( C \).

The author has shown\(^6\) that there exists a space \( B_1 \) of dimension \( n - 1 \), and an at most 2-to-1 mapping \( \phi \) of \( B_1 \) onto \( A \). Furthermore the subset of \( B_1 \) consisting of all \( x \) such that \( \phi^{-1}(\phi(x)) = x \) is \((n - 1)\)-dimensional. For \( y \in A \), if \( \phi^{-1}(y) \) is a single point then write \( f(y) = \phi^{-1}(y) \). If \( \phi^{-1}(y) \) is double-valued, select arbitrarily one point in \( \phi^{-1}(y) \) and define \( f(y) \) to be this point. Let \( f(A) = B \). Then \( B \) is of dimension \( n - 1 \) and \( f \), as inverse of a 1-1 continuous function, is open.

Remark. If \( f(A) = B \) is open and 1-1, then \( f^{-1} \) is continuous and 1-1. Thus Theorem 3 provides examples of arbitrary increases of dimension by 1-1 mappings.

THEOREM 4. Suppose \( f(A) = B \) is open, \( B \) is locally compact, and for each \( y \in B \) the set \( f^{-1}(y) \) is not dense-in-itself. Then \( \dim B \leq \dim A \).\(^7\)

PROOF. Let \( K_1, K_2, \ldots \) be an open base for \( A \). For each \( i \) let \( A_i \) be the set of all \( x \in K_i \) such that if \( x' \in K_i \) and \( f(x') = f(x) \), then \( x' = x \), and let \( B_i = f(A_i) \). For \( y \in B \), \( f^{-1}(y) \) contains an isolated point (with respect to \( f^{-1}(y) \)), so \( B = \bigcup_{i=1}^{\infty} B_i \). We prove next that \( B_i \) is closed in the open set \( f(K_i) \). For suppose on the contrary that there exists a sequence \( y_n \rightarrow y \), where \( y \) and each \( y_n \) are in \( f(K_i) \), \( y_n \in B_i \) but \( y \notin B_i \). Since \( y \in f(K_i) \) but not in \( B_i \), there exist distinct points \( x \) and \( x' \) in \( K_i \) such that \( f(x) = f(x') = y \). For each \( n \) there is a unique \( x_n \in K_i \) such that \( f(x_n) = y_n \). Since \( y_n \rightarrow y \) it follows that \( \lim \inf f^{-1}(y_n) \supset f^{-1}(y) \supset (x+x') \). But \( f^{-1}(y_n) \) has only the one point \( x_n \) in \( K \). This gives a contradiction.

Now let \( M_{i_1}, M_{i_2}, \ldots \) be closed and compact subsets of \( B \) with \( M_{i_1} + M_{i_2} + \cdots = f(K_i) \). Write \( B_i = B_i \cdot M_{i_2} \). Then \( B_i \) is a compact space and \( B = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} B_{i_2} \). Hence if \( \dim B_{i_2} \leq \dim A \) for all \( i \) and \( j \), then \( \dim B \leq \dim A \) (see \([4, \text{Theorem III 2, p. 30}]\)). Write \( A_{i_2} = A_i \cdot f^{-1}(B_{i_2}) \). Over \( A_{i_2} \), \( f \) is open and 1-1 to \( B_{i_2} \). For \( y \in B_{i_2} \) define \( g(y) = x \in A_{i_2} \) such that \( f(x) = y \). Now \( g \), as the inverse of any open 1-1 transformation, is continuous; thus \( g \) is a mapping and \( g(B_{i_2}) = A_{i_2} \). Furthermore \( B_{i_2} \) is compact, so that \( g \) is a closed mapping. Also \( g \) is 1-1. Then\(^8\) \( \dim B_{i_2} \leq \dim A \), so \( \dim B \leq \dim A \).

Remark. In Theorem 4 the assumption that \( B \) is locally compact

\(^6\) See \([3]\), especially Theorem 9.1. If the \( M \) of this theorem is the space \( A \), then \( M_1 \) is the desired space \( B_1 \) and \( \phi_1 \) is the desired mapping \( \phi \).

\(^7\) See footnote 4.

\(^8\) See \([4, \text{Theorem VI 7, p. 91}]\).
can be replaced by the weaker assumption that every point of $B$ has arbitrarily small neighborhoods with compact boundary. The proof is not given. Some assumption of compactness seems necessary. Consider the following example.

**Example.** There exist, in the plane, spaces $A$ and $B$ and an open 1-1 transformation $f$ with $f(A) = B$, with $\dim A = 0$ and $\dim B = 1$.

The space $B$ is an example due to Sierpiński [6, pp. 81–83]. This space $B$ has the following properties: (1) it is 1-dimensional, (2) it lies in the plane with $0 \leq x \leq 1$, $0 \leq y \leq 1$, (3) it contains at most one point $(x, y)$ for a given $x$, and (4) the set of all points $(x, 0)$, such that for some $y$ the point $(x, y) \in B$, is homeomorphic to the Cantor ternary set. Let $A$ be this set defined by (4). Then $A$ is the projection of $B$ onto the $x$-axis, and for $(x, 0) \in A$ there is a single point $(x, y)$ in $B$. Define $f$ as follows: $f(x, 0) = f(x, y) \in B$. Then $f$ has the required properties.

**BIBLIOGRAPHY**


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