REMARKS ON METRIZABILITY

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In connection with those paragraphs of my paper *Applications of the theory of Boolean rings to general topology* (Trans. Amer. Math. Soc. vol. 41 (1937) pp. 375–481) dealing with regular spaces, I have long been curious to know whether certain results proved there could be used to obtain the well known theorem that a separable (Hausdorff) space is metrizable if (and only if) it is regular. Since a positive answer to the question thus posed may have some interest from a methodological point of view, I communicate a demonstration here. The essential step in this demonstration even has some intrinsic interest, consisting as it does in the proof of new facts about dissection-spaces and the related maps. However, as a proof of the metrizability theorem this discussion is not as simple or as direct as the now classical proof of Tychonoff and Urysohn—which, it may be recalled, consists in showing, first, that a separable regular space is normal[1] and, second, that a separable normal space is metrizable.[2]

As a direct corollary of theorems established in our paper cited above, we may state the following result.

THEOREM. *If $\mathfrak{R}$ is a separable regular space, then $\mathfrak{R}$ has a map $m(\mathfrak{R}, \mathfrak{S}, \mathfrak{X})$ where $\mathfrak{X}$ is a continuous family of disjoint closed sets in a compact metric space $\mathfrak{S}$, which may be taken as a closed subset of the Cantor discontinuum; in other words, $\mathfrak{R}$ is topologically equivalent to the space obtained by introducing the "weak" topology in $\mathfrak{X}$.*

Theorems 26 and 69 of our paper show that the desired map can be constructed with $\mathfrak{S}$ taken to be the Boolean space representing the countable Boolean algebra generated by an arbitrarily chosen countable basis for $\mathfrak{R}$; but Theorems 1, 10, and 13 show that the space $\mathfrak{S}$ is topologically equivalent to a closed subspace of the Cantor discontinuum.

We shall now establish the following result.

THEOREM. *Let $\mathfrak{X}$ be a continuous family[3] of mutually disjoint, non-void, compact subsets $\mathfrak{x}$ in a metric space $\mathfrak{S}$. Then the space obtained by*

Presented to the Society, November 30, 1946; received by the editors October 17, 1946.

[1] Tychonoff, Math. Ann. vol. 95 (1925–1926) pp. 139–142.

[2] Urysohn, Math. Ann. vol. 94 (1925) pp. 309–315.

[3] In the terminology of R. L. Moore, an upper semi-continuous collection.

introduction of the "weak" topology in $\mathfrak{X}$ is metrizable; *and, if $\mathfrak{S}$ is separable, so is $\mathfrak{X}$.*

Since the final statement of the theorem is an immediate corollary of Theorem 21 of our cited paper, we need concern ourselves only with the metrizability of $\mathfrak{X}$.

Now the system of neighborhoods assigned in the "weak" topology to an arbitrary $\mathfrak{X}_0$ in $\mathfrak{X}$ consists of all the subfamilies $\mathfrak{U}(\mathfrak{X}_0, \mathfrak{G})$ of $\mathfrak{X}$ obtainable by the following construction: $\mathfrak{G}$ is chosen arbitrarily as an open neighborhood of $\mathfrak{X}_0$—that is, as an open set in $\mathfrak{S}$ which contains $\mathfrak{X}_0$—and $\mathfrak{U}(\mathfrak{X}_0, \mathfrak{G})$ is then defined as the class of all $\mathfrak{X}$ in $\mathfrak{X}$ such that $\mathfrak{X}\subset\mathfrak{G}$. Our first step is to show that this system of neighborhoods can be replaced by an equivalent system of neighborhoods defined in terms of the distance-function ρ given in $\mathfrak{S}$. For this purpose, let $\mathfrak{G}(\mathfrak{X}_0, e)$, where $e>0$, denote the e-neighborhood of $\mathfrak{X}_0$—that is to say, the open set characterized by the fact that each of its points is at distance less than e from some point of $\mathfrak{X}_0$. It is now almost immediate that the neighborhoods $\mathfrak{U}(\mathfrak{X}_0, e)=\mathfrak{U}(\mathfrak{X}_0, \mathfrak{G}(\mathfrak{X}_0, e))$, where e runs through any sequence of positive values with limit 0, constitute a system contained in and equivalent to the system originally given. Indeed, the only detail to be verified is that each open neighborhood $\mathfrak{G}$ of $\mathfrak{X}_0$ determines a positive e_0 with the property that $\mathfrak{G}(\mathfrak{X}_0, e)\subset\mathfrak{G}$ for $e<e_0$. If the distance $\rho(\mathfrak{X}_0, \mathfrak{G}')$ between $\mathfrak{X}_0$ and the complement $\mathfrak{G}'$ of $\mathfrak{G}$ is not equal to 0, we can obviously take $e_0=\rho(\mathfrak{X}_0, \mathfrak{G}')$. Hence we can obtain the result desired if we can deduce a contradiction from the assumption that $\rho(\mathfrak{X}_0, \mathfrak{G}')=0$. This assumption implies the existence of sequences $\{\mathfrak{x}_n\}$ in $\mathfrak{X}_0$ and $\{\mathfrak{y}_n\}$ in $\mathfrak{G}'$ with $\lim_{n\to\infty}\rho(\mathfrak{x}_n, \mathfrak{y}_n)=0$, where by virtue of the compactness of $\mathfrak{X}_0$ the first sequence can be chosen so as to have a limit $\mathfrak{x}$ in $\mathfrak{X}_0$; but then the sequence $\{\mathfrak{y}_n\}$ must also have $\mathfrak{x}$ as its limit and, in view of the fact that $\mathfrak{G}'$ is closed, this limit is in $\mathfrak{G}'$. However, it is clear that there is no $\mathfrak{x}$ common to $\mathfrak{X}_0$ and $\mathfrak{G}'$.

The next step is to interpret in terms of these neighborhoods the hypothesis that the family $\mathfrak{X}$ is continuous. In terms of the original system of neighborhoods, the continuity of $\mathfrak{X}$ means that to every open neighborhood $\mathfrak{G}$ of $\mathfrak{X}_0$ there corresponds a second open neighborhood $\mathfrak{G}_0$ with the property that every $\mathfrak{X}$ having a point in common with $\mathfrak{G}_0$ is contained in $\mathfrak{G}$. If $\mathfrak{G}$ is here taken to be an e-neighborhood, then $e_0=\phi(\mathfrak{X}_0, e)<e$ can obviously be so chosen that the e_0-neighborhood of $\mathfrak{X}_0$ has the property required of $\mathfrak{G}_0$: for, if any suitable $\mathfrak{G}_0$ is found corresponding to $\mathfrak{G}$, we may choose e_0 so small that $\mathfrak{G}(\mathfrak{X}_0, e_0)\subset\mathfrak{G}_0$ and can then replace $\mathfrak{G}_0$ by $\mathfrak{G}(\mathfrak{X}_0, e_0)$. Since $\mathfrak{X}$ and $\mathfrak{X}_0$ are both compact the distance $\rho(\mathfrak{X}, \mathfrak{X}_0)$ is equal to the distance between two

points suitably chosen in the respective sets; and, accordingly, $\mathfrak{X}$ has a point in common with $\mathfrak{S}(\mathfrak{X}_0, e_0)$ if and only if $\rho(\mathfrak{X}, \mathfrak{X}_0) < e_0$. Thus the continuity of $\mathfrak{X}$ is seen to imply the existence of a real function $\phi(\mathfrak{X}_0, e)$ with the property that $\mathfrak{X} \in \mathfrak{U}(\mathfrak{X}_0, e)$ whenever $\rho(\mathfrak{X}, \mathfrak{X}_0) < \phi(\mathfrak{X}_0, e)$ and the property that $0 < \phi(\mathfrak{X}_0, e) < e$.

With each $\mathfrak{X}_0$ we can now associate a system of neighborhoods $\mathfrak{U}_n(\mathfrak{X}_0) = \mathfrak{U}(\mathfrak{X}_0, e_n)$ and a system of positive integers $m_n = m(\mathfrak{X}_0, n)$ so that whenever $\mathfrak{U}_m(\mathfrak{X}_0)$ and $\mathfrak{U}_m(\mathfrak{X})$, $m = m(\mathfrak{X}_0, n)$, have an element in common the relation $\mathfrak{U}_m(\mathfrak{X}) \subset \mathfrak{U}_n(\mathfrak{X}_0)$ is satisfied. In order to do so we put $e_1(\mathfrak{X}_0) = 1$, $e_{n+1}(\mathfrak{X}_0) = \phi(\mathfrak{X}_0, 2^{-1}e_n(\mathfrak{X}_0))$, $\mathfrak{U}_n(\mathfrak{X}_0) = \mathfrak{U}(\mathfrak{X}_0, e_n(\mathfrak{X}_0))$ and choose $m_n = m(\mathfrak{X}_0, n)$ as the least positive integer such that $2^{-m} < e_{n+2}(\mathfrak{X}_0)$. We obviously have, by definition and recursion, the inequalities $e_{n+1}(\mathfrak{X}_0) < 2^{-1}e_n(\mathfrak{X}_0)$, $e_n(\mathfrak{X}_0) < 2^{-n}$, and hence conclude that $m_n > n+2$. The system of neighborhoods $\mathfrak{U}_n(\mathfrak{X}_0)$ is, as we proved above, equivalent to that originally given in $\mathfrak{X}$. Now let $\mathfrak{X}_1$ be an element common to $\mathfrak{U}_m(\mathfrak{X}_0)$ and $\mathfrak{U}_m(\mathfrak{X})$, where $m = m(\mathfrak{X}_0, n)$; and let $\mathfrak{X}_2$ be an *arbitrary* element of $\mathfrak{U}_m(\mathfrak{X})$. If $\mathfrak{x}_1$ is an arbitrary point of $\mathfrak{X}_1$, there exist points $\mathfrak{x}_0$ and $\mathfrak{x}$ in $\mathfrak{X}_0$ and $\mathfrak{X}$ respectively such that $\rho(\mathfrak{x}_0, \mathfrak{x}_1) < e_m(\mathfrak{X}_0)$, $\rho(\mathfrak{x}, \mathfrak{x}_1) < e_m(\mathfrak{X})$. Hence we see that

$$\begin{aligned} \rho(\mathfrak{X}_0, \mathfrak{X}) \leqq \rho(\mathfrak{x}_0, \mathfrak{x}) &< e_m(\mathfrak{X}_0) + e_m(\mathfrak{X}) \\ &< 2 \cdot 2^{-m} < 2e_{n+2}(\mathfrak{X}_0) \\ &< e_{n+1}(\mathfrak{X}_0) = \phi(\mathfrak{X}_0, 2^{-1}e_n(\mathfrak{X}_0)). \end{aligned}$$

Consequently we see that $\mathfrak{X} \in \mathfrak{U}(\mathfrak{X}_0, 2^{-1}e_n(\mathfrak{X}_0))$. Now let $\mathfrak{x}_2$ be an arbitrary point of $\mathfrak{X}_2$. Since $\mathfrak{X}_2$ is in $\mathfrak{U}_m(\mathfrak{X})$, there exists a point $\mathfrak{x}$ in $\mathfrak{X}$ such that $\rho(\mathfrak{x}_2, \mathfrak{x}) < e_m(\mathfrak{X}) < 2^{-1}e_n(\mathfrak{X}_0)$; and since $\mathfrak{X}$ is in $\mathfrak{U}(\mathfrak{X}_0, 2^{-1}e_n(\mathfrak{X}_0))$ there exists a point $\mathfrak{x}_0$ in $\mathfrak{X}_0$ such that $\rho(\mathfrak{x}, \mathfrak{x}_0) < 2^{-1}e_n(\mathfrak{X}_0)$. Hence we have $\rho(\mathfrak{x}_2, \mathfrak{x}_0) < e_n(\mathfrak{X}_0)$, $\mathfrak{X}_2 \in \mathfrak{U}_n(\mathfrak{X}_0)$, and $\mathfrak{U}_m(\mathfrak{X}) \subset \mathfrak{U}_n(\mathfrak{X}_0)$.

The result of the preceding paragraph permits the application of a theorem of A. H. Frink,[4] according to which the space $\mathfrak{X}$ is metrizable.

Combining the two theorems above, we evidently obtain the following result.

THEOREM. *A separable regular space $\mathfrak{R}$ is metrizable.*

For the space $\mathfrak{S}$ of the first theorem and its closed subsets are compact; and the second theorem is therefore applicable.

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[4] A. H. Frink, Bull. Amer. Math. Soc. vol. 43 (1937) pp. 133–142, Theorem 3.