SOME REMARKS ON THE THEORY OF GRAPHS

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The present note consists of some remarks on graphs. A graph G is a set of points some of which are connected by edges. We assume here that no two points are connected by more than one edge. The complementary graph G' of G has the same vertices as G and two points are connected in G' if and only if they are not connected in G.

A special case of a theorem of Ramsey can be stated in graph theoretic language as follows:

There exists a function $f(k, l)$ of positive integers k, l with the following property. Let there be given a graph G of $n \geqq f(k, l)$ vertices. Then either G contains a complete graph of order k, or G' a complete graph of order l. (A complete graph is a graph any two vertices of which are connected. The order of a complete graph is the number of its vertices.)

It would be desirable to have a formula for $f(k, l)$. This at present we can not do. We have however the following estimates:

THEOREM I. *Let* $k \geqq 3$. *Then*

$$2^{k/2} < f(k, k) \leqq C_{2k-2,k-1} < 4^{k-1}.$$

The second inequality of Theorem I was proved by Szekeres,[1] thus we only consider the first one. Let $N \leqq 2^{k/2}$. Clearly the number of different graphs of N vertices equals $2^{N(N-1)/2}$. (We consider the vertices of the graph as distinguishable.) The number of different graphs containing a given complete graph of order k is clearly $2^{N(N-1)/2}/2^{k(k-1)/2}$. Thus the number of graphs of $N \leqq 2^{k/2}$ vertices containing a complete graph of order k is less than

$$(1) \qquad C_{N,k}\frac{2^{N(N-1)/2}}{2^{k(k-1)/2}} < \frac{N^k}{k!}\frac{2^{N(N-1)/2}}{2^{k(k-1)/2}} < \frac{2^{N(N-1)/2}}{2}$$

since by a simple calculation for $N \leqq 2^{k/2}$ and $k \geqq 3$

$$2N^k < k!2^{k(k-1)/2}.$$

But it follows immediately from (1) that there exists a graph such that neither it nor its complementary graph contains a complete subgraph of order k, which completes the proof of Theorem I.

The following formulation of Theorem I might be of some interest:

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[1] P. Erdös and G. Szekeres, Compositio Math. vol. 2 (1935) pp. 463–470.

Define $A(n)$ as the greatest integer such that given any graph G of n vertices, either it or its complementary graph contains a complete subgraph of order $A(n)$. Then for $A(n) \geqq 3$

$$\frac{\log n}{2 \log 2} < A(n) < \frac{2 \log n}{\log 2} \cdot \tag{2}$$

The proof of (2) follows immediately from Theorem I. ($4^{A(n)} > n$, $2^{A(n)/2} < n$.)

The general theorem of Ramsey will now be stated.

THEOREM (RAMSEY). *Given three positive integers i, k, l, $i \leqq k$, $i \leqq l$, there exists a function $f(i, k, l)$ with the following property: If $n \geqq f(i, k, l)$ and if there is given a collection of combinations of order i of n elements, such that every combination of order k contains at least one combination of order i of the collection, then there exists a combination of order l all of whose combinations of order i belong to the collection.*

Several proofs of this theorem have been published.[1] Szekeres's proof gives the best known limits for $f(i, k, l)$. He proves[1]

$$f(i, k, l) \leqq f(i-1, f(i, k-1, l), f(i, k, l-1)) + 1; \tag{3}$$

also clearly $f(1, k, l) = k + l - 1$; $f(i, i, l) = l$; $f(i, k, i) = k$. By the same method we used in the proof of Theorem I we obtain that for sufficiently large k

$$f(i, k, k) > c^{k^{i-1}/i!}, \tag{4}$$

or

$$A(i, n) < c_1(\log n)^{1/(i-1)}.$$

(To see this put $k = c_1(\log n)^{1/(i-1)}$ in (4) and observe $f(i, k, k) > n$, for sufficiently large c_1.) Here c and c_1 depend only on n and i, and $A(i, n)$ is the greatest integer with the following property: Split the combinations of order k of n elements into two classes U_1 and U_2 in an arbitrary way. Then there exist $A(i, n)$ elements all whose combinations of order k are either in U_1 or in U_2. The values given by (4) are very much smaller than the values given by (3).

From (3) we obtain[1]

$$f(k, l) = f(2, k, l) \leqq C_{k+l-2,k}.$$

Thus

$$f(3, l) \leqq C_{l+1,2}.$$

It is possible that

$$f(3, l) = O(l).$$

Our method used in the proof of Theorem I does not enable us to show that $f(3, l)/l \to \infty$.

Before concluding we prove the following theorem.

THEOREM II. *Let there be given $(k-1)(l-1)+1$ integers $a_1 < a_2 < \cdots$. Then either there exist k of them no one dividing the other or l of them each a multiple of the previous one.*

Construct a matrix $a_i^{(j)}$ with the following properties: (1) no $a_i^{(j)}$ is a multiple of any $a_i^{(r)}$ with $j \leqq r$; (2) every $a_i^{(r+1)}$ is a multiple of some $a_i^{(r)}$; (3) all the a's occur among the a_i^j once and only once. If any row contains k or more elements we have k a's, no one dividing the other. If not, it clearly follows that the number of rows must be at least l. Now we obtain from (2) that by considering any $a_i^{(l)}$ we obtain a sequence of l a's, each being a multiple of the previous one, which completes the proof. The $(k-1)(l-1)$ integers p_u^v, $1 \leqq u \leqq k-1$; $1 \leqq v \leqq l-1$, p_u primes, show that $(k-1)(l-1)+1$ is best possible.[2]

By the same method we can prove the following theorem.

THEOREM IIa. *Let there be given a graph G of $(k-1)(l-1)+1$ vertices. Then either G contains a complete graph of order k, or G' contains a directed path of l vertices, for every orientation of the edges of G' in which there are no directed closed paths.*

Recently very much more general theorems have been proved by Grünwald and Milgram.[3] They in fact proved (among others) that the condition that G' contains no closed directed path is superfluous.

We suppress the proof of Theorem IIa since it is essentially the same as that of Theorem II. (We only remark that a connected to b by a line directed from a to b should be replaced by a divides b.)

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[2] This proof is due to J. Brunings.

[3] Oral communication.