ON NÖRLUND SUMMABILITY OF RANDOM VARIABLES TO ZERO

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1. **Introduction.** In a previous paper[1] [**1**], the author considered the Cesàro summability methods $\{\boldsymbol{C}_r\}$ $(0<r<\infty)$ for sequences of independent, real-valued random variables $\{\mathbf{x}_k\}$. For summability in probability of $\{\mathbf{x}_k\}$ to 0, it was shown that: (i) $r<s$ implies $\boldsymbol{C}_r \subset \boldsymbol{C}_s$; (ii) for $r\geqq 1$ all the methods $\boldsymbol{C}_r$ are essentially equivalent, in contrast to the Cesàro theory for sequences of real numbers. The field of the investigation reported here is the summability in probability of sequences $\{\mathbf{x}_k\}$ to 0 by the Nörlund summability methods, which include the Cesàro methods. The objective (attained only in special cases) was to prove that if two Nörlund methods N_p and N_q share the relation $N_p \subset N_q$ over sequences of real numbers, then the analogous relation $\boldsymbol{N}_p \subset \boldsymbol{N}_q$ holds for the summability of sequences of independent, real-valued, symmetric random variables to zero. The converse is, of course, false.

The only sequences $\{\mathbf{x}_k\}$ considered here are normal families of independent, real-valued, symmetric random variables. For these $\{\mathbf{x}_k\}$ the objective has been attained for three special cases; see Theorems 4, 5, and 6. The earlier theorems are tools: Theorem 1 gives a necessary and sufficient condition for the Nörlund summability of $\{\mathbf{x}_k\}$ to 0, while Theorems 2 and 3 give sufficient conditions for the relations $\boldsymbol{N}_p \subset \boldsymbol{N}_q$ and $\boldsymbol{N}_p \equiv \boldsymbol{N}_q$, respectively. Theorem 7 shows that equivalence with $\boldsymbol{C}_1$ over $\{\mathbf{x}_k\}$ extends to a Nörlund method $\boldsymbol{N}_p$ whose counterpart N_p is strictly weaker than C_1 over sequences of real numbers. Such equivalence with $\boldsymbol{C}_1$ is impossible for Cesàro methods weaker than C_1 over sequences of real numbers.

It is conjectured that Theorems 4, 5, and 6, here proved for normal families only, can be extended without change of statement to arbitrary sequences of independent, real-valued, symmetric random variables. If the $\mathbf{x}_k$ are not symmetric there are complications (see [**1**]), but it is conjectured that Theorems 4, 5, and 6 still hold without essential change.

2. **Nörlund summability of sequences of real numbers.** Let $p=\{p_n\}$ $(n=0, 1, 2, \cdots)$ be a sequence of nonnegative real numbers, with

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[1] Numbers in brackets refer to the references cited at the end of the paper.

$p_0=1$; for each n let $P_n=\sum_0^n p_k$. For any sequence $\{x_k\}$ of real numbers, a transformed sequence $\{y_n\}$ is defined by the relation $y_n=P_n^{-1}\sum_{k=0}^n p_{n-k}x_k$ $(n=0, 1, 2, \cdots)$. If the sequence $\{y_n\}$ has the limit x', the sequence $\{x_k\}$ is said to be summable-N_p to x', where N_p is the Nörlund summability method corresponding to p. The method N_p is known to be regular (that is, consistent with ordinary convergence, for all convergent sequences $\{x_k\}$) if and only if $p_n=o(P_n)$. There is a substantial known theory of the Nörlund summability methods (see [**2, 6, 7, 8, 9**]), of which certain results will be quoted in this section for comparison with their analogues for sequences of random variables.

For two summability methods A and B, the statement "$A\subset B$" means that any sequence summable-A to a finite limit is also summable-B to the same limit. The statement "$A\equiv B$" means that $A\subset B$ and $B\subset A$. The negation of "$A\subset B$" is "$A\not\subset B$."

In addition to N_p, defined by $\{p_k\}$, let a second Nörlund summability method N_q be defined by $\{q_k\}$, with $q_0=1$, $q_k\geqq 0$, and $Q_n=\sum_0^n q_k$. The following generating functions and coefficients are defined formally by M. Riesz [**7**]:

$$p^*(x) = \sum_{n=0}^{\infty} p_n x^n; \qquad q^*(x) = \sum_{n=0}^{\infty} q_n x^n;$$

$$\lambda^*(x) = \frac{q^*(x)}{p^*(x)} = \sum_{n=0}^{\infty} \lambda_n^* x^n;$$

$$\mu^*(x) = \frac{p^*(x)}{q^*(x)} = \sum_{n=0}^{\infty} \mu_n^* x^n.$$

It is assumed in (2-1) to (2-5) that N_p and N_q are both regular.

(2-1) (M. Riesz [**7**]) *$N_p\subset N_q$ if and only if, as $n\to\infty$, both $\sum_{k=0}^n P_k|\lambda_{n-k}^*|=O(Q_n)$ and $\lambda_n^*=o(Q_n)$.*

(2-2) (M. Riesz [**7**]) *$N_p\equiv N_q$ if and only if $\sum_{n=0}^{\infty}(|\lambda_n^*|+|\mu_n^*|)<\infty$.*

The Cesàro summability methods C_r $(0<r<\infty)$ are of the Nörlund type N_p, where $p^*(x)=(1-x)^{-r}$. If we let N_p be C_1, (2-2) takes the following form ([**2**], p. 782):

(2-3) *$C_1\subset N_q$ if and only if, as $n\to\infty$, $\sum_{k=0}^n(n+1-k)|q_k-q_{k-1}|=O(Q_n)$, where $q_{-1}=0$.*

It follows immediately from (2-3) that:

(2-4) *If $q_n\leqq q_{n+1}$ for all n, then $C_1\subset N_q$.*

It follows from (2-1) that:

(2-5) *If* $p_n \geqq p_{n+1}$ *for all* n, *then* $N_p \subset C_1$.

The author has not seen (2-5) in the literature, but it is undoubtedly known. It can be proved much like the analogous Theorem 6 of the next section.

3. **Nörlund summability of random variables to zero.** Sections 3 and 4 of [1] contain the notation, basic definitions, and references for the summability in probability of a sequence $\{\mathbf{x}_k\}$ to 0. We shall here confine ourselves to sequences of independent, real-valued, symmetric random variables which form a normal family.

DEFINITION 1 (P. Lévy [5]). $\{\mathbf{x}_k\}$ forms a *normal family* when both (3-1) and (3-2) hold:

(3-1) $E(\mathbf{x}_k)=0$ *and* $E(\mathbf{x}_k^2)=\sigma_k^2<\infty$, *for all* k;

(3-2) *There exists a random variable* $\mathbf{x}$ *with finite* $E(\mathbf{x}^2)$ *such that for all* $A>0$ *and all* k

$$\text{Prob}\left\{ |\mathbf{x}_k| > A\sigma_k \right\} \leqq \text{Prob}\left\{ |\mathbf{x}| > A \right\}.$$

(Lévy does not require that $E(\mathbf{x}_k)=0$.)

DEFINITION 2. With the notation of §2 above, a sequence $\{\mathbf{x}_k\}$ is said to be *summable-N_p in probability to* 0 when for each $\epsilon>0$, as $n\to\infty$,

$$\text{Prob}\left\{ \left| P_n^{-1} \sum_{k=0}^{n} p_{n-k}\mathbf{x}_k \right| > \epsilon \right\} \to 0. \tag{3-3}$$

The words "in probability" will sometimes be omitted for brevity.

The basic theorem is the following:

THEOREM 1. $\{\mathbf{x}_k\}$ *is a normal family of independent, real-valued, symmetric random variables. In order that* $\{\mathbf{x}_k\}$ *be summable-N_p in probability to* 0, *it is necessary and sufficient that, as* $n\to\infty$,

$$\sum_{k=0}^{n} p_{n-k}^2\sigma_k^2 = o(P_n^2). \tag{3-4}$$

Theorem 1 follows immediately from Theorem 5.3 of [1] by letting $p_{n-k}P_n^{-1}=a_{nk}$. The proof of Theorem 5.3 in [1] did not depend on the special character of $\|a_{nk}\|$ as a Cesàro matrix.

The basic tools of the present investigation are Theorems 2 and 3, which are analogues of (2-1) and (2-2). In the following, regularity of N_p for sequences of real numbers is not assumed unless explicitly stated. It is assumed always that $p_n \geqq 0$, $q_n \geqq 0$, $p_0 = q_0 = 1$.

Let the following generating functions and coefficients be defined formally:

$$p(x) = \sum_{n=0}^{\infty} p_n^2 x^n; \qquad q(x) = \sum_{n=0}^{\infty} q_n^2 x^n;$$

$$\lambda(x) = \frac{q(x)}{p(x)} = \sum_{n=0}^{\infty} \lambda_n x^n; \qquad \mu(x) = \frac{p(x)}{q(x)} = \sum_{n=0}^{\infty} \mu_n x^n.$$

THEOREM 2. *Statements* (3-5) *and* (3-6) *together form a sufficient condition for* $N_p \subset N_q$ *with respect to the summability in probability of normal families of independent, real-valued, symmetric random variables to* 0:

$$\sum_{k=0}^{n} P_k^2 \left| \lambda_{n-k} \right| = O(Q_n^2); \tag{3-5}$$

$$\lambda_n = o(Q_n^2). \tag{3-6}$$

PROOF. Let $\{\mathbf{x}_k\}$ be any normal family of independent, real-valued, symmetric random variables which is summable-N_p to 0. Suppose (3-5) and (3-6) hold. Let $t_k = P_k^{-2}\sum_{h=0}^{k} p_{k-h}^2 \sigma_h^2$ ($k = 0, 1, 2, \cdots$). By Theorem 1, $\{t_k\}$ is a null-sequence of non-negative numbers. Let $u_n = Q_n^{-2}\sum_{k=0}^{n} q_{n-k}^2 \sigma_k^2$ ($n = 0, 1, 2, \cdots$). To prove Theorem 2 it is sufficient to show that $\{u_n\}$ is also a null-sequence and apply Theorem 1 again.

Define $\sigma(x)$ as $\sum_{k=0}^{\infty}\sigma_k^2 x^k$, formally. Now $t_n P_n^2 = \sum_{k=0}^{n} p_{n-k}^2 \sigma_k^2$, by definition of t_n. Hence $\sigma(x)p(x) = \sum_{n=0}^{\infty} t_n P_n^2 x^n$. Similarly, $\sigma(x)q(x) = \sum_{n=0}^{\infty} u_n Q_n^2 x^n$. But $\sigma(x)q(x) = [\sigma(x)p(x)]\lambda(x)$. Hence, by equating coefficients, we see that we may write $u_n = \sum_{k=0}^{n} b_{nk} t_k$, where $b_{nk} = Q_n^{-2} P_k^2 \lambda_{n-k}$. The sequence $\{u_n\}$ is seen to be obtained as the transform of $\{t_k\}$ by the triangular matrix $\|b_{nk}\|$. To prove Theorem 2 it suffices to show that $\|b_{nk}\|$ is null-preserving. By a theorem of Kojima [**4**], $\|b_{nk}\|$ is null-preserving if and only if:

$$\lim_{n\to\infty} b_{nk} = 0, \qquad \text{for each } k; \tag{3-7}$$

$$\sum_{k=0}^{n} \left| b_{nk} \right| \leqq M < \infty, \qquad \text{for all } n. \tag{3-8}$$

But (3-6) is equivalent to (3-7), and (3-5) is equivalent to (3-8). This proves Theorem 2.

Since its members are non-negative, $\{t_k\}$ is not an arbitrary null-sequence; hence this proof cannot yield necessary conditions for

$N_p \subset N_q$. Indeed, (3-5) is not necessary; see Corollary 2, following Theorem 7.

The symbol C_r $(r \geqq 0)$ represents the Cesàro summability method of order r over sequences of random variables.

COROLLARY 1. *Let N_q be a regular Nörlund summability method. Set $q_{-1}=0$. A sufficient condition for $C_1 \subset N_q$ with respect to the summability in probability of normal families of independent, real-valued, symmetric random variables to* 0 *is that*

$$\sum_{k=0}^{n} (n-k+1)^2 \left| q_k^2 - q_{k-1}^2 \right| = O(Q_n^2). \tag{3-9}$$

PROOF. Identify C_1 with N_p in Theorem 2. It is found that $p(x)=(1-x)^{-1}$ and that $\lambda_n = q_n^2 - q_{n-1}^2$. Condition (3-6) is automatically satisfied, since N_q is regular. Condition (3-5) is thus sufficient for $C_1 \subset N_q$. But when N_p is C_1 ,(3–5) takes the form (3-9), proving the corollary.

THEOREM 3. *Suppose $p_n \leqq p_{n+1}$ and $q_n \leqq q_{n+1}$, for all n. Then a sufficient condition that $N_p \equiv N_q$ with respect to the summability in probability to* 0 *of normal families of independent, real-valued, symmetric random variables is that*

$$\sum_{k=0}^{\infty} (|\lambda_k| + |\mu_k|) < \infty. \tag{3-10}$$

PROOF. Let $\sum_{k=0}^{\infty} |\lambda_k| = A^2 < \infty$ and $\sum_{k=0}^{\infty} |\mu_k| = B^2 < \infty$. Since $p(x) = q(x)\mu(x)$, and since q_n is nondecreasing, we find that $p_n^2 = \sum_{k=0}^{n} q_k^2 \mu_{n-k} \leqq q_n^2 \sum_{k=0}^{n} |\mu_{n-k}| \leqq B^2 q_n^2$. Thus $p_n \leqq B q_n$, whence it follows that, for all $k \leqq n$, $P_k \leqq B Q_n$. Hence, for all n,

$$Q_n^{-2} \sum_{k=0}^{n} P_k^2 |\lambda_{n-k}| \leqq B^2 \sum_{k=0}^{n} |\lambda_{n-k}| \leqq B^2 A^2 < \infty.$$

Therefore (3-5) is satisfied. Since $\sum |\lambda_n| < \infty$, $\lambda_n = o(1)$ and (3-6) is satisfied. By Theorem 2, $N_p \subset N_q$ for all normal families under consideration. By interchanging p and q it is seen similarly that $N_q \subset N_p$. Hence $N_p \equiv N_q$, proving Theorem 3.

With the strong hypothesis that p_n and q_n are nondecreasing, Theorem 3 is rather weak. In fact, it follows from Theorem 5 that if $p_n \leqq p_{n+1}$ and $q_n \leqq q_{n+1}$, for all n, *and if N_p and N_q are both regular*, then $N_p \equiv N_q$.

THEOREM 4. *Let N_q be any regular Nörlund summability method such* *that $C_1 \subset N_q$ with respect to sequences of real numbers. Then $\mathbf{C}_1 \subset \boldsymbol{N}_q$ with respect to the summability in probability of normal families of independent, real-valued, symmetric random variables to* 0.

PROOF. If N_q satisfies the hypothesis, we know by (2-3) that

$$\sum_{k=0}^{n} (n + 1 - k) \left| q_k - q_{k-1} \right| = O(Q_n). \tag{3-11}$$

By Lemma 3, proved in §4, (3-11) implies (3-9). By Corollary 1 above, (3-9) implies that $\boldsymbol{C}_1 \subset \boldsymbol{N}_q$.

Theorem 5 is a second special case in which it is shown that $N_p \subset N_q$ implies $\boldsymbol{N}_p \subset \boldsymbol{N}_q$. Indeed, in this case it is even shown that $\boldsymbol{N}_p \equiv \boldsymbol{N}_q$. Since it is possible that $N_q \not\subset N_p$, it is untrue that $\boldsymbol{N}_q \subset \boldsymbol{N}_p$ implies $N_q \subset N_p$.

THEOREM 5. *Let N_q be any regular Nörlund summability method such that $q_n \leqq q_{n+1}$, for all n. Then $\mathbf{C}_1 \equiv \boldsymbol{N}_q$ with respect to the summability in probability of normal families of independent, real-valued, symmetric random variables to* 0.

PROOF. Given a regular N_q with $q_n \leqq q_{n+1}$ for all n. By (2-4), $C_1 \subset N_q$. Hence, by Theorem 4, $\boldsymbol{C}_1 \subset \boldsymbol{N}_q$. There remains only the proof that $\boldsymbol{N}_q \subset \boldsymbol{C}_1$. This will be given in two steps.

I. Since N_q is regular, for each n we have, as $r \to \infty$, $q_r/Q_{n+r} \leqq q_r/Q_r \to 0$. Let $\delta(n) = \max_{r \geqq 0}(q_r/Q_{n+r})$. We shall prove that

$$n\delta(n) \geqq \beta > 0 \qquad (n = 1, 2, 3, \cdots). \tag{3-12}$$

Let $t_r = q_r/Q_r$. Then

$$\frac{q_r}{Q_{n+r}} = t_r(1 - t_{r+1})(1 - t_{r+2}) \cdots (1 - t_{r+n}).$$

Fix $n > 0$. Since $t_0 = 1$ and $\lim_r t_r = 0$, we may let $r(n)$ be the largest value of r for which $t_r \geqq (n+1)^{-1}$. Then

$$\begin{aligned} \frac{q_{r(n)}}{Q_{n+r(n)}} &> \frac{1}{n+1}\left(1 - \frac{1}{n+1}\right)^n \\ &= \frac{1}{n+1}\left\{\left(1 + \frac{1}{n}\right)^n\right\}^{-1} \cong \frac{e^{-1}}{n+1}. \end{aligned} \tag{3-13}$$

Since $\delta(n) \geqq q_{r(n)}/Q_{n+r(n)}$, it is seen from (3-13) that (3-12) is true.

II. Suppose, for a contrapositive proof, that $\{\mathbf{x}_k\}$ is a normal family of independent, real-valued, symmetric random variables which is not summable-$\boldsymbol{C}_1$ to 0. Let $s_{-1} = 0$; for $n \geqq 0$ let $s_n = \sum_{k=0}^{n} \sigma_k^2 \leqq s_{n+1}$. By

Theorem 1, $\{n^{-2}s_n\}$ is not a null-sequence; we can therefore find $\epsilon>0$ and integers $0<n_1<n_2<\cdots<n_i<n_{i+1}<\cdots$ such that

$$s_{n_i} \geqq \epsilon n_i^2 \qquad (i = 1, 2, \cdots). \tag{3-14}$$

Let $q_{-1}=0$. Let $\phi(n)=Q_n^{-2}\sum_{k=0}^n \sigma_k^2 q_{n-k}^2 = Q_n^{-2}\sum_{k=0}^n s_k(q_{n-k}^2-q_{n-k-1}^2)$. By Theorem 1, $\{\mathbf{x}_k\}$ is summable-$\boldsymbol{N}_q$ to 0 if and only if $\{\phi(n)\}$ is a null-sequence. To complete the contrapositive proof we show that $\{\phi(n)\}$ is not a null-sequence. Fix any n_i. Since q_n and s_n are both nondecreasing, it is seen for all $n\geqq n_i$ that

$$\begin{aligned}\phi(n) &\geqq Q_n^{-2}\sum_{k=n_i}^{n} s_k(q_{n-k}^2 - q_{n-k-1}^2)\\ &\geqq Q_n^{-2}s_{n_i}\sum_{k=n_i}^{n}(q_{n-k}^2 - q_{n-k-1}^2)\\ &= s_{n_i}q_{n-n_i}^2 Q_n^{-2}\\ &\geqq \epsilon n_i^2 q_{n-n_i}^2 Q_n^{-2}.\end{aligned}$$

The last inequality is by (3-14). Hence for some integer n_i' greater than n_i we have

$$\phi(n_i') > \epsilon n_i^2\left\{2^{-1}\max_{n\geqq n_i}(q_{n-n_i}^2 Q_n^{-2})\right\} = 2^{-1}\epsilon n_i^2\{\delta(n_i)\}^2.$$

By (3-12) we see that $\phi(n_i')>2^{-1}\epsilon\beta^2>0$. Thus $\{\phi(n)\}$ cannot be a null-sequence, completing the proof of Theorem 5.

The satisfaction of condition (3-12) is equivalent to regularity for Nörlund summability methods N_p whose counterparts $\boldsymbol{N}_p$ can sum to 0 a sequence $\{\mathbf{x}_k\}$ of random variables which are not all identically zero. That is, nonregular Nörlund methods which satisfy (3-12) have only a trivial applicability to the summability of random variables.

It was proved in [**1**, Theorem 5.10] that, for $r>1$, $\boldsymbol{C}_1\equiv\boldsymbol{C}_r$ with respect to the summability to 0 of *arbitrary* sequences $\{\mathbf{x}_k\}$ of independent, real-valued, symmetric random variables. When we further restrict $\{\mathbf{x}_k\}$ to normal families, the present Theorem 5 extends identity with $\boldsymbol{C}_1$ to a wide class of Nörlund summability methods including all methods $\boldsymbol{C}_r$ for $r>1$.

Theorem 6 is the third special case in which it is shown that $N_p\subset N_q$ implies $\boldsymbol{N}_p\subset\boldsymbol{N}_q$.

THEOREM 6. *Let N_p be any Nörlund summability method such that $p_n\geqq p_{n+1}$, for all n. Then $\boldsymbol{N}_p\subset\boldsymbol{C}_1$ with respect to the summability in prob-*

ability of normal families of independent, real-valued, symmetric random variables to 0.

PROOF. Let C_1 be identified with the N_q of Theorem 2. We have $p(x)=\sum_{n=0}^{\infty}p_n^2x^n$ and $q(x)=(1-x)^{-1}$. Letting $a_n=p_{n-1}^2-p_n^2$, we see formally that $\sum_{n=0}^{\infty}\lambda_nx^n=q(x)/p(x)=(1-\sum_{n=1}^{\infty}a_nx^n)^{-1}$. Hence

$$\left(\sum_{n=0}^{\infty}\lambda_nx^n\right)\left(1-\sum_{n=1}^{\infty}a_nx^n\right)=1. \tag{3-15}$$

Equating coefficients in (3-15), we find that $\lambda_0=1$ and

$$\lambda_n=a_1\lambda_{n-1}+a_2\lambda_{n-2}+\cdots+\lambda_0a_n \qquad (n=1,2,\cdots). \tag{3-16}$$

Since $\lambda_0=1$ and since $a_n\geqq 0$, it is seen from (3-16) that $\lambda_n\geqq 0$ for all n. (The non-negativity of $\{\lambda_n\}$ is also a result of Kaluza [3].) Now $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}(p_{n-1}^2-p_n^2)=p_0^2=1$. By (3-16), λ_n is a weighted sum of $\{\lambda_0,\lambda_1,\cdots,\lambda_n\}$, with total weight $\sum_{k=1}^{n}a_k\leqq 1$. Therefore $0\leqq\lambda_n\leqq\max_{0\leqq k\leqq n-1}\lambda_k$. Since $\lambda_0=1$, this implies that all $\lambda_n\leqq 1$. But $Q_n=n+1$. Hence $\lambda_n=o(Q_n^2)$, proving (3-6).

Let $R_n=\sum_{k=0}^{n}p_k^2$. We have the following formal identities:

$$\lambda(x)=\frac{q(x)}{p(x)}=\frac{(1-x)^{-1}q(x)}{(1-x)^{-1}p(x)}=\frac{(1-x)^{-2}}{(1-x)^{-1}\sum_n p_n^2x^n}=\frac{\sum_n(n+1)x^n}{\sum_n R_nx^n}.$$

Therefore $\lambda(x)\sum_{n=0}^{\infty}R_nx^n=\sum_{n=0}^{\infty}(n+1)x^n$, formally. Equating coefficients of x^n and remembering that $\lambda_n\geqq 0$, we find that

$$\sum_{k=0}^{n}R_k\left|\lambda_{n-k}\right|=n+1 \qquad (n=0,1,2,\cdots). \tag{3-17}$$

But, by Schwarz's inequality, $P_k^2=(\sum_{j=0}^{k}p_j)^2\leqq(\sum_{j=0}^{k}p_j^2)(\sum_{j=0}^{k}1)=(k+1)R_k$. Hence

$$P_k^2\leqq(n+1)R_k \qquad (k=0,1,2,\cdots,n). \tag{3-18}$$

From (3-17) and (3-18) it is seen that $\sum_{k=0}^{n}P_k^2\left|\lambda_{n-k}\right|\leqq(n+1)\sum_{k=0}^{n}R_k\left|\lambda_{n-k}\right|=(n+1)^2=Q_n^2$. Hence (3-5) is satisfied. By Theorem 2 the present proof is complete.

Any Nörlund method N_p with nonincreasing p_n is necessarily regular, for $p_n/P_n\leqq p_n/np_n=1/n\to 0$, as $n\to\infty$. Moreover, $N_p\subset C_1$.

The Cesàro summability methods C_r $(0<r<1)$ are of the Nörlund type with nonincreasing $\{p_n\}$. Theorem 5.5 of [1] showed that, for $0<r<1$, $C_r\subset C_1$ over *arbitrary* sequences of independent, real-valued, symmetric random variables. Theorem 5.6 of [1] showed that, for $0<r<1$, $\mathbf{C}_1 \not\subset \mathbf{C}_r$, even over *normal families*. Hence the conclusion of Theorem 6 cannot in general be extended to assert that $\mathbf{N}_p \equiv \mathbf{C}_1$.

One may ask whether Theorem 4 has a converse. That is, if N_q is regular, and if $C_1 \not\subset N_q$ over real sequences, can one always find a normal family summable-$\mathbf{C}_1$ to 0 but not summable-$\mathbf{N}_q$ to 0? When N_q is C_r $(0<r<1)$, we just saw that the answer is "yes." In general, however, the answer is "no," as is shown by the following theorem.

THEOREM 7. *There exists a regular Nörlund summability method* N_p *with the following properties*:

(3-19) $N_p \subset C_1$, *with respect to sequences of real numbers*;

(3-20) $C_1 \not\subset N_p$, *with respect to sequences of real numbers*;

(3-21) $\mathbf{N}_p \equiv \mathbf{C}_1$, *with respect to the summability in probability of normal families of independent, real-valued, symmetric random variables to* 0.

PROOF. Let $p_n = 1$ (n even); let $p_n = 0$ (n odd). Then $P_n \cong n/2$. Now $p^*(x) = \sum_n p_n x^n = \sum_n p_n^2 x^n = p(x) = (1-x^2)^{-1}$. Let C_1 be identified with the N_q of (2-1), (2-2), and Theorem 2. Then $q_n = 1$ and $Q_n = n+1$, for all n. Furthermore, $q^*(x) = \sum_n q_n x^n = \sum_n q_n^2 x^n = (1-x)^{-1}$. Then $\lambda^*(x) = \lambda(x) = (1-x)^{-1}/(1-x^2)^{-1} = 1+x$, while $\mu^*(x) = \mu(x) = 1/\lambda(x) = (1+x)^{-1}$. Hence $\lambda_0^* = \lambda_0 = \lambda_1^* = \lambda_1 = 1$; $\lambda_n^* = \lambda_n = 0$ $(n \geqq 2)$; $\mu_n^* = \mu_n = (-1)^n$, for all n.

I. Obviously $\lambda_n^* = o(Q_n)$; and $\sum_{k=0}^n P_k \left|\lambda_{n-k}^*\right| = P_n + P_{n-1} \cong n = O(Q_n)$. Hence, by (2-1), $N_p \subset C_1$ over sequences of real numbers.

II. Since $\sum_{n=0}^\infty \left|\mu_n^*\right| = \infty$, we find from (2-2) that $N_p \not\equiv C_1$ for sequences of real numbers. Since $N_p \subset C_1$, we know that $C_1 \not\subset N_p$.

III. Obviously $\lambda_n = o(Q_n^2)$; and $\sum_{k=0}^n P_k^2 \left|\lambda_{n-k}\right| = P_n^2 + P_{n-1}^2 \cong n^2/2 = O(Q_n^2)$. Hence, by Theorem 2, $\mathbf{N}_p \subset \mathbf{C}_1$ for the $\{\mathbf{x}_k\}$ of (3-19).

IV. To prove $\mathbf{C}_1 \subset \mathbf{N}_p$ we are unable to use Theorem 2 by interchanging p and q and putting $\{\mu_n\}$ into (3-5) and (3-6). However, (3-5) is not a necessary condition, and we shall show directly that $\mathbf{C}_1 \subset \mathbf{N}_p$. Consider any normal family $\{\mathbf{x}_k\}$ which is summable-$\mathbf{C}_1$ to 0. By Theorem 1, $n^{-2}\sum_{k=0}^n \sigma_k^2 \to 0$, as $n \to \infty$. Hence $P_n^{-2}\sum_{k=0}^n p_{n-k}^2 \sigma_k^2 \leqq P_n^{-2}\sum_{k=0}^n \sigma_k^2 \cong (4/n^2)\sum_{k=0}^n \sigma_k^2 \to 0$, as $n \to \infty$. Theorem 1, applied to $\mathbf{N}_p$, proves that $\{\mathbf{x}_k\}$ is summable-$\mathbf{N}_p$ to 0. Hence $\mathbf{C}_1 \subset \mathbf{N}_p$ and so, by III above, $\mathbf{C}_1 \equiv \mathbf{N}_p$. This completes the proof of Theorem 7.

It is curious that the necessity of condition (3-5) for $\mathbf{N}_p \subset \mathbf{N}_q$ over normal families $\{\mathbf{x}_k\}$ breaks down just for an example where we fail to have $N_p \subset N_q$ over real sequences.

COROLLARY 2. *Condition* (3-5) *is not necessary for* $\mathbf{N}_p \subset \mathbf{N}_q$ *with re-*

spect to the summability in probability of a normal family of independent, real-valued, symmetric random variables to 0.

4. **Lemmas used in proof of Theorem** 4. The following Lemmas 1 and 2 are used to prove Lemma 3, which was applied to the proof of Theorem 4. In all three lemmas, $c_{-1}=0$ and $\{c_i\}$ $(i=0, 1, 2, \cdots)$ is an arbitrary sequence of non-negative numbers. The number $|c_i-c_{i-1}|$ is usually abbrevated to h_i.

LEMMA 1. *With the above notation, for each integer* $n=1, 2, 3, \cdots$, (4-1) *holds*:

$$2\sum_{i=0}^{n-1}\left| c_i^2 - c_{i-1}^2 \right| \leqq 2\sum_{i,j=0}^{n-1} h_ih_j. \tag{4-1}$$

PROOF. We use a proof by induction. Since $c_{-1}=0$, (4-1) is true for $n=1$. Suppose that (4-1) holds for $n=N$. Let $c_N \geqq 0$ be arbitrary. Now

$$\begin{aligned}\sum_{i=0}^{N}\left| c_i^2 - c_{i-1}^2 \right| &\leqq \left| c_N^2 - c_{N-1}^2 \right| + \sum_{i,j=0}^{N-1} h_ih_j \\ &= \left| c_N^2 - c_{N-1}^2 \right| + \sum_{i,j=0}^{N} h_ih_j - 2\sum_{i=0}^{N-1} h_ih_N - h_N^2.\end{aligned}$$

Thus to prove (4-1) for $n=N+1$ it is sufficient to prove that

$$\left| c_N^2 - c_{N-1}^2 \right| - 2\sum_{i=0}^{N-1} h_ih_N - h_N^2 \leqq 0. \tag{4-2}$$

Now $c_{N-1}=\sum_{i=0}^{N-1}(c_i-c_{i-1}) \leqq \sum_{i=0}^{N-1}|c_i-c_{i-1}|$. Therefore

$$2c_{N-1} \leqq 2\sum_{i=0}^{N-1} h_i. \tag{4-3}$$

Case 1. Suppose $c_N \geqq c_{N-1}$. Adding $c_N-c_{N-1}=h_N$ to both sides of (4-3), we find that

$$c_N + c_{N-1} \leqq 2\sum_{i=0}^{N-1} h_i + h_N. \tag{4-4}$$

Multiplying both sides of (4.3) by $|c_N-c_{N-1}|=h_N$, we see that (4-2) holds, proving the lemma for Case 1.

Case 2. Suppose $c_N<c_{N-1}$. Then from (4-3) we see that $2c_N<2\sum_{i=0}^{N-1}h_i$. Adding $c_{N-1}-c_N=h_N$ to both sides of the last inequality, we see that (4-4) holds. The proof of the lemma is completed as in Case 1.

LEMMA 2. *With the same notation, for each integer* $n=1, 2, 3, \cdots$, (4-5) *holds*:

$$2\sum_{i=0}^{n-1} \left| c_i^2 - c_{i-1}^2 \right| + \left| c_n^2 - c_{n-1}^2 \right| \leqq 2\sum_{i,j=0}^{n-1} h_i h_j + 2h_n \sum_{i=0}^{n-1} (n+1-i)h_i + h_n^2. \tag{4-5}$$

PROOF. By Lemma 1 it suffices to prove that

$$\left| c_n^2 - c_{n-1}^2 \right| \leqq 2h_n \sum_{i=0}^{n-1} (n+1-i)h_i + h_n^2. \tag{4-6}$$

If $h_n = |c_n - c_{n-1}| = 0$, (4-6) is trivial. If not, (4-6) is equivalent to (4-7):

$$c_n + c_{n-1} \leqq 2\sum_{i=0}^{n-1} (n+1-i)h_i + h_n. \tag{4-7}$$

To prove (4-7), we start with the inequality

$$\begin{aligned} c_{n-1} &\leqq c_0 + c_1 + \cdots + c_{n-2} + 2c_{n-1} \\ &= \sum_{i=0}^{n-1} (n+1-i)(c_i - c_{i-1}). \end{aligned}$$

Hence

$$2c_{n-1} \leqq 2\sum_{i=0}^{n-1} (n+1-i)h_i. \tag{4-8}$$

Now (4-7) follows from (4-8) just as (4-4) followed from (4-3), by use of two cases. Thus the proof is complete.

LEMMA 3. *With the same notation, for each integer* $n = 0, 1, 2, \cdots$, (4-9) *holds*:

$$\sum_{k=0}^{n} (n-k+1)^2 \left| c_k^2 - c_{k-1}^2 \right| \leqq \left\{ \sum_{k=0}^{n} (n-k+1) \left| c_k - c_{k-1} \right| \right\}^2. \tag{4-9}$$

PROOF. Let $\phi_n = \sum_{k=0}^{n}(n-k+1)^2 |c_k^2 - c_{k-1}^2|$, for $n = 0, 1, 2, \cdots$. Let $\phi_{-1} = 0$. Let $\psi_n = \{\sum_{k=0}^{n}(n-k+1)h_k\}^2 = \sum_{i,j=0}^{n}(n-i+1)(n-j+1)h_i h_j$, for $n = 0, 1, 2, \cdots$. Let $\psi_{-1} = 0$. We must prove that $\phi_n \leqq \psi_n$, for all n. Since the result is trivial for $n = 0$, fix $n \geqq 1$.

We take first and second differences of ϕ_n and ψ_n:

$$\Delta\phi_n = \phi_n - \phi_{n-1} = \sum_{k=0}^{n} (2n - 2k + 1) \left| c_k^2 - c_{k-1}^2 \right|.$$

$$\Delta^2\phi_n = \Delta\phi_n - \Delta\phi_{n-1} = 2\sum_{k=0}^{n-1} \left| c_k^2 - c_{k-1}^2 \right| + \left| c_n^2 - c_{n-1}^2 \right|.$$

$$\Delta\psi_n = \psi_n - \psi_{n-1} = \sum_{i,j=0}^{n} (2n + 1 - i - j) h_i h_j.$$

$$\Delta^2\psi_n = \Delta\psi_n - \Delta\psi_{n-1} = 2\sum_{i,j=0}^{n-1} h_i h_j + 2h_n \sum_{i=0}^{n-1} (n + 1 - i) h_i + h_n^2.$$

Now by Lemma 2, $\Delta^2\phi_n \leqq \Delta^2\psi_n$ for all integers $n \geqq 1$. Hence $\Delta\phi_n \leqq \Delta\psi_n$ and therefore $\phi_n \leqq \psi_n$ for the same integers. This proves Lemma 3.

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