ON THE STRUCTURE OF INTRINSIC DERIVATIVES

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Introduction. The primary purpose of the present paper is to express the Mth order intrinsic derivative of a higher order absolute tensor such as T^{ab}, T_{ab}, or T^{ab}_{cd} as a contraction of extensors. As a first step, we develop a rule for constructing extensors of the types $E^{\alpha a \cdot \beta b}$, $E_{\alpha a \cdot \beta b}$, $E^{\alpha a : \beta b}_{\gamma c \cdot \delta d}$ from absolute tensors T^{ab}, T_{ab}, T^{ab}_{cd} by repeated differentiation with respect to the curve parameter followed by multiplication by the appropriate coefficients. We then consider the contractions of the various E's with the extended components of connection $L^{e}_{\alpha a}$, $L^{\alpha a}_{e}$ (to be introduced) and prove by induction that these contractions give the Mth order intrinsic derivatives of the original tensors. In this way we establish a highly satisfactory theory of the algebraic structure of the higher order intrinsic derivatives, for the constituent E's and L's obviously possess an invariance of form and being extensors they are such that other extensors, tensors, and invariants can be built from them and other extensors by simple algebraic procedures—addition, multiplication, and contraction.

1. **Notation and preliminaries.** In the present paper we shall employ at most two coordinate systems x and $\bar{x}$ and so far as the quantities that bear indices are concerned, we shall distinguish between them whenever feasible by restricting the choice of indicial letters. Specifically, letters at the first of the alphabet a, b, c, d, e shall serve to denote system x, while r, s, t, u, v, w will be correlated to system $\bar{x}$. Thus x^r is the rth coordinate variable of system $\bar{x}$, while x^a is variable number a of system x. Differentiation with respect to the parameter t of a parameterized arc will be indicated by primes and Greek indices, the latter are enclosed except in certain abridged symbols. To illustrate,

$$x'^a = x^{a'} = dx^a/dt, \qquad x^{(\alpha)a} = x^{a(\alpha)} = d^\alpha x^a/dt^\alpha,$$
$$X^{\rho r}_{\alpha\rho} = X^{(a)r}_{(\alpha)a} = \partial x^{(\rho)r}/\partial x^{(\alpha)a}, \qquad X^{\alpha a}_{\rho r} = \partial x^{(\alpha)a}/\partial x^{(\rho)r}.$$

Furthermore, we assume that there is given an affine connection L^a_{bc} and let L^a_b, called the two-index connection, represent $L^a_{bc}x'^c$.

Summation convention. Repeated lower case Latin indices call for summations 1 to N, while the summations indicated by repeated lower case Greek indices are from zero (unless the contrary is specified) to some terminal value usually M or M+1. Repeated capital Greek in-

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dices do not generate sums, thus $\binom{M}{\alpha} U_a^{\cdot(M-\alpha)}$ with α not summed would be written $\binom{M}{A} U_a^{\cdot(M-A)}$.

Extensors. Extensors will be denoted by means of symbols bearing Greek-Latin doublet indices such as ρr or αa. The extensor transformation law is exemplified by the equation

$$T^{\rho r}_{\sigma s \cdot t} = T^{\alpha a}_{\beta b \cdot c} X^{\rho r}_{\alpha a} X^{\beta b}_{\sigma s} X^{c}_{t}.$$

This particular extensor, or more properly tensor-extensor since the component symbols bear a tensor index, may be described by saying it is of the type (1, 1, 0, 1, 0)—the numbers referring to the number of doublet superscripts, doublet subscripts, single superscripts, single subscripts, and the weight. Here the range of the Greek indices is 0 to **M**; however, in the summations this is effectively reduced since X sup ρr inf αa vanishes whenever α exceeds ρ.

Owing to the algebraic character of the extended coordinate transformation so far as the x''s are concerned, extensors have many properties not held by tensors. In particular there are $\mathbf{M}+1$ contractions instead of one, and a greater variety of quotient laws. In the present paper we shall have need for but two of these laws.

Quotient laws. Let $V^{\alpha a}$ and $U_{\alpha a}$ be defined by

$$V^{\alpha a} = V^{a(\alpha)}, \qquad U_{\alpha a} = \binom{M}{A} U_a^{(M-A)}, \qquad A = \alpha. \tag{1.1}$$

(1) If for arbitrary U, $U_{\alpha a} T^{\alpha a \cdots}_{\cdots}$ is an extensor of the type indicated by its free indices, then the same may be asserted of T.

(2) If $V^{\alpha a} T_{\alpha a \cdots}^{\cdots}$ is an extensor, then T is likewise an extensor assuming of course that V^a is arbitrary.

In a preceding note [**3**],[1] we have proved that if $T^c_{\alpha a}$ and $T^{\alpha a}_c$ are extensors of range 0 to M and of the type indicated by their indices, then their extensive derivatives $DT|^c_{\alpha a}$ and $D_1 T|^{\alpha a}_c$ defined by

$$DT|^c_{\alpha a} = T^c_{\alpha-1\cdot a} + T^{c\cdot\prime}_{\alpha a} + T^b_{\alpha a} L^c_b, \tag{1.2}$$

$$\begin{aligned} D_1 T|^{\alpha a}_c &= \frac{M+1-A}{M+1} T^{Aa}_c \\ &\quad + \frac{A}{M+1}\left(T^{A-1\cdot a\prime}_c - T^{A-1\cdot a}_b L^b_c\right), \qquad A = \alpha, \end{aligned} \tag{1.3}$$

are extensors of the extended range 0 to $\mathbf{M}+1$. The hair line is introduced to emphasize that the indices c and αa designate components

[1] Numbers in brackets refer to the references cited at the end of the paper.

of the derived extensor and not of the original. The subscript on D in (1.3) will serve to distinguish the two derivatives (upper and lower) in case $\mathrm{M}=0$ and consequently T is a tensor—a special case. The proper range of α on the left is 0 to $\mathrm{M}+1$, while the proper range of the first index on the right is 0 to M. The special statements needed for the cases $\alpha=0$ and $\alpha=\mathrm{M}+1$ will be contained in (1.2) and (1.3) if we adopt, as we now do, the following convention:

Any symbol bearing an indicial number outside of the proper range of the index in question is to be given the value zero. Thus, in particular, $T_{-1\cdot a}=T_{\mathrm{M}+1\cdot a}=0$.

The quantities obtained by repeatedly applying upper and lower extensive differentiation to the Kronecker delta δ^a_b will be called the extended components of connection. The result of contracting these components with the $V^{\alpha a}$ and $U_{\alpha a}$ defined in equation (1.1) is the Mth order intrinsic derivative [**3**] of V and U.

2. **The derived extensors.** Since the quantities $V^{\alpha a}$ and $U_{\alpha a}$ of (1.1) are extensors and are made up of the derivatives of V and U, it is quite obvious that one can develop extensors from higher order tensors by first contracting with V^a, U_a, and so on, differentiating the product, "factoring" out $V^{\alpha a}$, $U_{\alpha a}$, and so on, and then applying the quotient law. As a preliminary to this procedure, we introduce certain generalized binomial coefficients one of which at least—the multinomial coefficient—is well known. These quantities arise from applying the Leibnitz rule for differentiating a product and formula (1.1). To simplify the writing, we restrict ourselves to the specific definition of the typical special cases.

DEFINITION 2.1.

$$\binom{\mathrm{M}}{\mathrm{A},\ \mathrm{B},\ \Gamma}=\frac{\mathrm{M}!}{\mathrm{A}!\mathrm{B}!\Gamma!(\mathrm{M}-\mathrm{A}-\mathrm{B}-\Gamma)!}, \qquad \mathrm{M}-\mathrm{A}-\mathrm{B}-\Gamma \geqq 0,$$

$$\binom{\mathrm{M}}{\mathrm{A},\ \mathrm{B},\ \Gamma}=0, \qquad \text{if } \mathrm{M}-\mathrm{A}-\mathrm{B}-\Gamma<0;$$

$$\begin{bmatrix}\mathrm{A},\ \mathrm{B},\ \Gamma\\ \mathrm{M}\end{bmatrix}=\binom{\mathrm{M}}{\mathrm{M}-\mathrm{A},\ \mathrm{M}-\mathrm{B},\ \mathrm{M}-\Gamma}\binom{\mathrm{M}}{\mathrm{A}}^{-1}\binom{\mathrm{M}}{\mathrm{B}}^{-1}\binom{\mathrm{M}}{\Gamma}^{-1},$$
$$\mathrm{A}+\mathrm{B}+\Gamma-2\mathrm{M}\geqq 0,$$

$$\begin{bmatrix}\mathrm{A},\ \mathrm{B},\ \Gamma\\ \mathrm{M}\end{bmatrix}=0, \qquad \text{if } \mathrm{A}+\mathrm{B}+\Gamma-2\mathrm{M}<0;$$

$$\left\{\begin{matrix} \mathrm{A, B} \\ \mathrm{M}, \Gamma, \Delta \end{matrix}\right\} = \begin{bmatrix} \mathrm{A, B} \\ \mathrm{M} \end{bmatrix} \begin{pmatrix} \mathrm{A} + \mathrm{B} - \mathrm{M} \\ \Gamma, \Delta \end{pmatrix}$$

$$= \frac{\mathrm{A}!\mathrm{B}!}{\mathrm{M}!\Gamma!\Delta!(\mathrm{A} + \mathrm{B} - \mathrm{M} - \Gamma - \Delta)!}.$$

THEOREM 2.1. *If T^{abc} is a tensor of the type* (3, 0, 0) (*that is, three superscripts, no subscripts, and weight zero*) *and the necessary derivatives exist, then the quantities E defined by*

$$E^{\alpha a \cdot \beta b \cdot \gamma c \cdot} = \begin{bmatrix} \mathrm{A, B}, \Gamma \\ \mathrm{M} \end{bmatrix} T^{abc(\mathrm{A+B}+\Gamma-2\mathrm{M})}, \qquad \mathrm{A} = \alpha,\ \mathrm{B} = \beta,\ \Gamma = \gamma,$$

are the components of an extensor of the type (3, 0, 0, 0, 0).

PROOF. $(T^{abc}A_aB_bC_c)^{(\mathrm{M})}$ is an invariant for arbitrary choice of the covariant vectors A, B, C. Expanding this derivative by the Leibnitz rule for differentiating a multinomial and, in accordance with our convention, defining a symbol which bears a negative prime to be zero, we have, after dropping the indices a, b, c,

$$(TABC)^{(\mathrm{M})} = \sum \begin{pmatrix} \mathrm{M} \\ \alpha, \beta, \gamma \end{pmatrix} A^{(\alpha)}B^{(\beta)}C^{(\gamma)}T^{(\mathrm{M}-\alpha-\beta-\gamma)}; \qquad \alpha, \beta, \gamma{:}0 \text{ to } \mathrm{M}.$$

We now replace the dummy indices α, β, γ with $\mathrm{M}-\bar{\alpha}$, $\mathrm{M}-\bar{\beta}$, $\mathrm{M}-\bar{\gamma}$, drop the bars, and express A sup $(\mathrm{M}-\alpha)$ in terms of $A_{\alpha a}$ by (1.1). The result after introducing definition (2.1) and the definition of E is

$$(T^{abc}A_aB_bC_c)^{(\mathrm{M})} = E^{\alpha a \cdot \beta b \cdot \gamma c}A_{\alpha a}B_{\beta b}C_{\gamma c}$$

and our theorem follows from the quotient law.

Remark. From the proof, we see that in the generalization of this theorem to the case in which T is of contravariant order q, the multiplier 2 in the expression $-2\mathrm{M}$ which occurs in the definition of E would be replaced with $q-1$, thus

$$\begin{bmatrix} \mathrm{A, B}, \Gamma, \cdots \\ \mathrm{M} \end{bmatrix} T^{abc\cdots(\mathrm{A+B}+\Gamma\cdots-(q-1)\mathrm{M})}.$$

This situation arises because we replaced α with $\mathrm{M}-\alpha$, and so on.

THEOREM 2.2. *If T^{ab}_{cd} is a tensor of the type* (2, 2, 0), *then the quantities E defined by*

$$E^{\alpha a \cdot \beta b}_{\gamma c \cdot \delta d} = \left\{\begin{matrix} \mathrm{A, B} \\ \mathrm{M}, \Gamma, \Delta \end{matrix}\right\} T^{ab(\mathrm{A+B-M}-\Gamma-\Delta}_{cd} \qquad \mathrm{A} = \alpha,\ \mathrm{B} = \beta,\ \Gamma = \gamma,\ \Delta = \delta,$$

are the components of an extensor of the type (2, 2, 0, 0, 0).

PROOF. Taking q to be two in the generalization of Theorem 2.1, it follows that for arbitrary U and V, the quantities

$$\begin{bmatrix} \mathrm{A, B} \\ \mathrm{M} \end{bmatrix} (T^{ab}_{cd} U^c V^d)^{(\mathrm{A+B-M})}$$

constitute the components of an extensor of the type (2, 0, 0, 0, 0). But

$$\begin{aligned} \begin{bmatrix} \mathrm{A, B} \\ M \end{bmatrix} & (T^{ab}_{cd} U^c V^d)^{(\mathrm{A+B-M})} \\ &= \sum_{\gamma,\delta=0} \begin{bmatrix} \mathrm{A, B} \\ \mathrm{M} \end{bmatrix} \begin{pmatrix} \mathrm{A + B - M} \\ \gamma, \delta \end{pmatrix} T^{ab(\mathrm{A+B-M}-\gamma-\delta)}_{cd} U^{c(\gamma)} V^{d(\delta)} \\ &= E^{\alpha a \cdot \beta b}_{\gamma c \cdot \delta d} U^{c(\gamma)} V^{d(\delta)} \end{aligned}$$

and the theorem follows from the quotient law. With regard to the range of indices it should be borne in mind that the proper range for $\alpha, \beta, \gamma, \delta$ is zero to M. Confining α and β to this range, we note in addition that $[{}^{\mathrm{A,B}}_{\mathrm{M}}]$ is zero if $\mathrm{A+B<M}$, thus the effective range of $\mathrm{A+B-M}$ is 0 to M inclusive. Consequently, the maximum range of γ and δ is 0 to M and this may be taken as the actual range for whenever $\gamma+\delta>\mathrm{A+B-M}$, the corresponding terms of the sum vanish.

Remark. If T were contravariant of unspecified order q, E would be defined by

$$E^{\alpha a \cdot \beta b \cdots}_{\gamma c \cdot \delta d} = \begin{Bmatrix} \mathrm{A, B, \cdots} \\ \mathrm{M, \Gamma, \Delta} \end{Bmatrix} T^{ab \cdots [\mathrm{A+B} \cdots - (q-1)\mathrm{M} - \Gamma - \Delta]}_{cd}.$$

THEOREM 2.3. *If* T_{cd} *is a tensor of the type* (0, 2, 0), *then the quantities E defined by*

$$E_{\gamma c \cdot \delta d} = \begin{pmatrix} \mathrm{M} \\ \Gamma, \Delta \end{pmatrix} T^{\cdot\cdot(\mathrm{M}-\Gamma-\Delta)}_{cd}$$

are the components of an extensor of the type (0, 2, 0, 0, 0).

PROOF. In the proof of the preceding theorem delete the indices a, b and the symbol $[{}^{\mathrm{A,B}}_{\mathrm{M}}]$ and replace $(\mathrm{A+B-M})$ with M. The theorem then follows from the quotient law.

Remark. The generalization of this theorem to higher order tensors is obvious.

3. **Recursion formulas.** To facilitate later developments, we list

here some recursion formulas for the E's and L's. Specifically, these relationships express the E's and L's and their derivatives for a given range M in terms of the corresponding quantities for range M+1. The higher range M+1 will be indicated by means of a star. To illustrate the notation and method, we develop the first formula in detail. In the right members of these formulas the capitals A, B, Γ, and Δ are equal to α, β, γ, δ, respectively, but do not generate sums. From the definition of E sup $\alpha a\cdot\beta b$, we have

$$\begin{aligned} E^{\alpha a\cdot\beta b} &= \begin{bmatrix}\mathrm{A},\ \mathrm{B}\\ \mathrm{M}\end{bmatrix} T^{ab(\mathrm{A}+\mathrm{B}-\mathrm{M})} \\ &= \begin{bmatrix}\mathrm{A},\ \mathrm{B}\\ \mathrm{M}\end{bmatrix}\begin{bmatrix}\mathrm{A}+1,\ \mathrm{B}\\ \mathrm{M}+1\end{bmatrix}^{-1}\begin{bmatrix}\mathrm{A}+1,\ \mathrm{B}\\ \mathrm{M}+1\end{bmatrix} T^{ab(\mathrm{A}+1+\mathrm{B}-\mathrm{M}-1)} \\ &= \begin{bmatrix}\mathrm{A},\ \mathrm{B}\\ \mathrm{M}\end{bmatrix}\begin{bmatrix}\mathrm{A}+1,\ \mathrm{B}\\ \mathrm{M}+1\end{bmatrix}^{-1} E^{*\mathrm{A}+1\cdot a\cdot\beta b} \\ &= \frac{\mathrm{M}+1}{\mathrm{A}+1} E^{*\mathrm{A}+1\cdot a\cdot\beta b}. \end{aligned}$$

Thus we have established

$$E^{\alpha a\cdot\beta b} = \frac{\mathrm{M}+1}{\mathrm{A}+1} E^{*\mathrm{A}+1\cdot a\beta b} = \frac{\mathrm{M}+1}{\mathrm{B}+1} E^{*\alpha a\cdot\mathrm{B}+1\cdot b}. \tag{3.1}$$

Similarly, it may be shown that

$$E^{\alpha a\cdot\beta b'} = \frac{(\mathrm{M}+1)(\mathrm{A}+\mathrm{B}-\mathrm{M}+1)}{(\mathrm{A}+1)(\mathrm{B}+1)} E^{*\mathrm{A}+1\cdot a\cdot\mathrm{B}+1\cdot b}, \tag{3.2}$$

$$E'_{\gamma c\cdot\delta d} = \frac{\mathrm{M}+1-\Gamma-\Delta}{\mathrm{M}+1} \overset{*}{E}_{\Gamma c\cdot\Delta d}, \tag{3.3}$$

$$E_{\gamma-1\cdot c\cdot\delta d} = \frac{\Gamma}{\mathrm{M}+1} \overset{*}{E}_{\Gamma c\cdot\delta d}, \qquad E_{\gamma c\cdot\delta-1\cdot d} = \frac{\Delta}{\mathrm{M}+1} \overset{*}{E}_{\gamma c\cdot\Delta d}, \tag{3.4}$$

$$E^{\alpha a\cdot\beta b}_{\gamma c\cdot\delta d} = \frac{\mathrm{M}+1}{\mathrm{A}+1} E^{*\mathrm{A}+1\cdot a\cdot\beta b}_{\gamma c\cdot\delta d} = \frac{\mathrm{M}+1}{\mathrm{B}+1} E^{*\alpha a\cdot\mathrm{B}+1\cdot b}_{\gamma c\cdot\delta d}, \tag{3.5}$$

$$E^{\alpha a\cdot\beta b}_{\gamma c\cdot\delta d} = \frac{(\mathrm{M}+1)(\Gamma+1)}{(\mathrm{A}+1)(\mathrm{B}+1)} E^{*\mathrm{A}+1\cdot a\mathrm{B}+1\cdot b}_{\Gamma+1\cdot c\cdot\delta d}, \tag{3.6}$$

$$E^{\alpha a\cdot\beta b'}_{\gamma c\cdot\delta d} = \frac{(\mathrm{M}+1)(\mathrm{A}+\mathrm{B}-\mathrm{M}-\Gamma-\Delta+1)}{(\mathrm{A}+1)(\mathrm{B}+1)} E^{*\mathrm{A}+1\cdot a\cdot\mathrm{B}+1\cdot b}_{\Gamma c\cdot\Delta d}. \tag{3.7}$$

The extended components of connection. As we have remarked before, the extended components of connection can be obtained from the Kronecker delta by applying repeatedly upper and lower extensive differentiation. In the case of upper extensive differentiation (that in which the superscript retains its tensor character) the Mth extensive derivative may be obtained by applying the formula: $D^{\mathrm{M}}\delta\big|^{c}_{\alpha a} = \binom{\mathrm{M}}{\mathrm{A}} I^{c}_{\mathrm{A}a}$ and Kawaguchi's recursion formula [4, p. 105, (13.6) and 3],

$$I^{c}_{\alpha-1\cdot a} = I^{c'}_{\alpha a} + I^{b}_{\alpha a} L^{c}_{b}.$$

Thus, if we let $L^{c}_{\alpha a}$ and $L^{*c}_{\alpha a}$ denote the extended components of connection for the cases M and M+1, then since $I^{c}_{\alpha a} = D^{\mathrm{M}}\delta\big|^{c}_{\alpha a}$ (by definition), we have the following table of values

$$L^{c}_{\mathrm{M}a} = \delta^{c}_{a}, \qquad L^{c}_{\mathrm{M}-1\cdot a} = \binom{\mathrm{M}}{\mathrm{M}-1} L^{c}_{a},$$

$$L^{c}_{\mathrm{M}-2\cdot a} = \binom{\mathrm{M}}{\mathrm{M}-2}\left(L^{c'}_{a} + L^{b}_{a} L^{c}_{b}\right),$$

and so on, and the obvious relationship,

$$L^{c}_{\mathrm{A}a}\binom{\mathrm{M}}{\mathrm{M}-\mathrm{A}}^{-1} = L^{*c}_{\mathrm{A}+1\cdot a}\binom{M+1}{\mathrm{M}-\mathrm{A}}^{-1}.$$

Thus we have

$$L^{c}_{\alpha a} = \frac{\mathrm{A}+1}{\mathrm{M}+1} L^{*c}_{\mathrm{A}+1\cdot a}. \tag{3.8}$$

Also, in geodesic coordinates Kawaguchi's formula may be written in the form $L^{c'}_{\alpha a} = \binom{\mathrm{M}}{\mathrm{A}}\binom{\mathrm{M}}{\mathrm{A}-1}^{-1} L^{c}_{\mathrm{A}-1}$ and this in conjunction with (3.8) yields

$$L^{c'}_{\alpha a} = \frac{\mathrm{M}+1-\mathrm{A}}{\mathrm{M}+1} L^{*c}_{\mathrm{A}a}. \tag{3.9}$$

Formula (3.9) is valid for substitution provided there is no subsequent differentiation.

Turning to the lower extensive derivative of the Kronecker delta (in this case the subscript retains its tensor character), the situation is somewhat simpler. The rule for the computation of this Mth derivative is as follows: Component upper $0c$ lower d of the Mth derivative of δ^{c}_{d} is given by $D_{\mathrm{M}}\delta\big|^{0c}_{d} = \delta^{c}_{d}$, the other components may be obtained from the recursion formula $D_{\mathrm{M}}\delta\big|^{\gamma c}_{d} = D_{\mathrm{M}}\delta\big|^{\gamma-1\cdot c'}_{d} - D_{\mathrm{M}}\delta\big|^{\gamma-1\cdot c}_{b} L^{b}_{d}$. Thus since we define $L^{\gamma c}_{d}$ to be $D_{\mathrm{M}}\delta\big|^{\gamma c}_{d}$, we have the following table for the extended components of connection, $L^{\gamma c}_{d}$:

$$L_d^{0c} = \delta_d^c, \qquad L_d^{1c} = -L_d^c, \qquad L_d^{2c} = -L_d^{c\prime} + L_b^c L_d^b,$$
$$L_d^{3c} = (-L_d^{c\prime} + L_b^c L_d^b)' - (-L_b^{c\prime} + L_e^c L_b^e) L_d^b,$$

and so on. Consequently, we have the equations,

$$L_d^{\gamma c} = L_d^{*\gamma c}, \qquad \text{for } \gamma < \mathrm{M} + 1; \tag{3.10}$$

$$L_d^{\gamma+1 \cdot c} = L_d^{\gamma c\prime} = L_d^{*\gamma+1 \cdot c}, \qquad \text{in geodesic coordinates.} \tag{3.11}$$

Equation (3.11) like equation (3.9) is valid for substitution only in case there is no subsequent differentiation.

4. **The higher intrinsic derivatives as contractions.** In case M is unity, we know from previous work [**2**, pp. 291–301] that a full contraction of any one of the E's with the L's yields the first order intrinsic derivative of the corresponding T. Accordingly, we turn our attention to the induction from M to $M+1$. Thus in the proofs that follow, we assume that a complete contraction of the unstarred quantities yields the Mth order intrinsic derivative and then attempt to establish the corresponding result for the starred quantities. Our first step is to compute the intrinsic derivative of the contraction of the unstarred quantities at the origin of a geodesic coordinate system and then, by means of formulas (3.1) through (3.11), express the result in terms of the starred quantities. With this in mind we are now ready to consider our final propositions.

THEOREM 4.1. *If $E_{\gamma c \cdot \delta d}$ denotes $\binom{\mathrm{M}}{\Gamma, \Delta} T_{cd}^{\cdot\cdot\,(\mathrm{M}-\Gamma-\Delta)}$, then the contraction $E_{\gamma c \cdot \delta d} L_a^{\gamma c} L_b^{\delta d}$ is the* M*th intrinsic derivative of the tensor T_{ab}.*

PROOF. If this contraction yields the Mth intrinsic derivative, then in geodesic coordinates the $(\mathrm{M}+1)$th derivative is given by

$$E'_{\gamma c \cdot \delta d} L_a^{\gamma c} L_b^{\delta d} + E_{\gamma c \cdot \delta d} L_a^{\gamma c\prime} L_b^{\delta d} + E_{\gamma c \cdot \delta d} L_a^{\gamma c} L_b^{\delta d\prime}.$$

By applying formulas (3.3) and (3.10), the first term transforms into

$$\sum_{\gamma, \delta = 0}^{\mathrm{M}+1} \frac{\mathrm{M} + 1 - \gamma - \delta}{\mathrm{M} + 1} E^*_{\gamma c \cdot \delta d} L_a^{*\gamma c} L_b^{*\delta d}. \tag{4.1}$$

The increase in the range of summation from M to M+1 is legitimate because the additional terms vanish since either the fractional coefficient or E^* vanishes if $\gamma + \delta \geqq \mathrm{M}+1$.

Similarly, the second term may be made to assume the form

$$\sum_{\gamma=0}^{\mathrm{M}+1} \frac{\gamma}{\mathrm{M} + 1} E^*_{\gamma c \cdot \delta d} L_a^{*\gamma c} L_b^{*\delta d} \tag{4.2}$$

by replacing γ with $\bar{\gamma}-1$, dropping the bar, and applying equations (3.4), (3.11), and (3.10). The lower range of γ should be one but the presence of the factor γ allows us to drop it to zero. The range of δ is 0 to M and of course this repeated index generates a sum. If we allow δ to take on the value $\mathrm{M}+1$, then the E^* would vanish unless γ has the value zero, but this value eliminates the term. Consequently, in (4.2) we may regard both of the indices γ and δ as generating sums over the range 0 to $\mathrm{M}+1$. Evidently, the third term may be treated similarly.

Obviously, the sum of the coefficients involved in the expressions (4.1), (4.2), and so on, is unity, and the theorem is established. Furthermore, it is quite apparent that this particular case is typical. The addition of more subscripts would not call for a change in procedure and the sum of the fractional coefficients would again be one.

Turning to the problem of mixed tensors, the question to be investigated may be formulated as follows:

THEOREM 4.2. *If* $E^{\alpha a \cdot \beta b}_{\gamma c \cdot \delta d}$ *denotes* $\left\{ {\mathrm{A}, \mathrm{B} \atop \mathrm{M}, \Gamma, \Delta} \right\} T^{ab(\mathrm{A}+\mathrm{B}-\mathrm{M}-\Gamma-\Delta)}_{cd}$ *then the contraction*

$$E^{\alpha a \cdot \beta b}_{\gamma c \cdot \delta d} L^{\;\;e}_{\alpha a} L^{\;\;f}_{\beta b} L^{\gamma c}_{\;\;g} L^{\delta d}_{\;\;h}$$

is the Mth *intrinsic derivative of the mixed tensor* T^{ef}_{gh}.

The method of verifying this statement is essentially that of Theorem 4.1. Accordingly, we may omit some of the explanation.

PROOF. The derivative of the contraction in question consists of five terms the first of which has a prime on the first E, the second a prime on the first L, and so on. By means of the relationships (3.7), (3.8), and (3.10), the first term may be written,

$$\frac{(\mathrm{M}+1)(\alpha+\beta-\mathrm{M}-\gamma-\delta+1)}{(\alpha+1)(\beta+1)} E^{*\alpha+1 \cdot a \cdot \beta+1 \cdot b}_{\gamma c \cdot \delta d} \frac{(\alpha+1)}{\mathrm{M}+1} \cdot L^{*\;\;e}_{\alpha+1 \cdot a} \frac{(\beta+1)}{\mathrm{M}+1} L^{*\;\;f}_{\beta+1 \cdot b} L^{*\gamma c}_{\;\;g} L^{*\delta d}_{\;\;h}.$$

Here there is summation on α, β, γ, δ from 0 to M. By replacing α and β with $\bar{\alpha}-1$ and $\bar{\beta}-1$, and subsequently dropping the bars, we get

$$(4.3) \qquad \frac{(\alpha+\beta-\mathrm{M}-\gamma-\delta-1)}{\mathrm{M}+1} E^{*\alpha a \cdot \beta b}_{\gamma c \cdot \delta d} L^{*\;\;e}_{\alpha a} L^{*\;\;f}_{\beta b} L^{*\gamma c}_{\;\;g} L^{*\delta d}_{\;\;h}.$$

It will be recalled that E^* vanishes whenever $\alpha+\beta<\mathrm{M}+1+\gamma+\delta$. Consequently, we may drop the range on α and β to zero and increase the range of γ and δ to $\mathrm{M}+1$. Thus the range of summation of α, β, γ, and δ in the expression (4.2) may be taken to be 0 to $\mathrm{M}+1$.

The second term of the derivative, that which contains the derivative of $L^{e}_{\alpha a}$, after application of the formulas (3.5), (3.9), (3.8), and (3.10) assumes the form

$$\frac{\mathrm{M}+1}{\beta+1} E^{*\alpha a\cdot\beta+1\cdot b}_{\gamma c\cdot\delta d} \frac{\mathrm{M}+1-\alpha}{\mathrm{M}+1} L^{*e}_{\alpha a} \frac{\beta+1}{\mathrm{M}+1} L^{*f}_{\beta+1\cdot b} L^{*\gamma c}_{g} L^{*\delta d}_{h}$$

or, replacing β with $\bar{\beta}-1$ and dropping the bar,

$$\frac{\mathrm{M}+1-\alpha}{\mathrm{M}+1} E^{*\alpha a\cdot\beta b}_{\gamma c\cdot\delta d} L^{*e}_{\alpha a} L^{*f}_{\beta b} L^{*\gamma c}_{g} L^{*\delta d}_{h}. \tag{4.4}$$

The summation on α, γ, and δ is 0 to M and on β is 1 to $\mathrm{M}+1$. We observe that if α is given the value $\mathrm{M}+1$, the coefficient vanishes and recall that if $\alpha+\beta$ is less than $\mathrm{M}+1+\gamma+\delta$ the E^* is zero. Consequently, if we assign β the value zero, α the value M, then $\alpha+\beta<\mathrm{M}+1+\gamma+\delta$. Therefore, α and β may each be given the range 0 to $\mathrm{M}+1$. Furthermore, if γ is given the value $\mathrm{M}+1$, α the value M, then again $\alpha+\beta<\mathrm{M}+1+\gamma+\delta$. Hence, we conclude that the range of α, β, γ, δ in (4.4) may be taken to be 0 to $\mathrm{M}+1$.

If we denote the product of the E with the L's that occurs in (4.4) by P, then the third term of the derivative in question (it contains $L^{f\prime}_{\beta b}$) may be transformed into the expression

$$\sum \frac{\mathrm{M}+1-\beta}{\mathrm{M}+1} P. \tag{4.5}$$

To treat the fourth and fifth terms, we apply equations (3.6), (3.8), (3.10), (3.11). The result is

$$\frac{(\mathrm{M}+1)(\gamma+1)}{(\alpha+1)(\beta+1)} E^{*\alpha+1\cdot a\cdot\beta+1\cdot b}_{\gamma+1\cdot c\cdot\delta d} \frac{\alpha+1}{\mathrm{M}+1} \frac{\beta+1}{\mathrm{M}+1} L^{*e}_{\alpha+1\cdot a} L^{*f}_{\beta+1\cdot b} L^{*\gamma+1\cdot c}_{g} L^{*\delta d}_{h},$$

which upon following previous procedures may be written

$$\sum \frac{\gamma}{\mathrm{M}+1} P. \tag{4.6}$$

Similarly, from the last term of the derivative, we get

$$\sum \frac{\delta}{\mathrm{M}+1} P. \tag{4.7}$$

The sum of the expressions (4.3) through (4.7) is

$$\sum \frac{\alpha+\beta-\mathrm{M}-\gamma-\delta-1+\mathrm{M}+1-\alpha+\mathrm{M}+1-\beta+\gamma+\delta}{\mathrm{M}+1} P \quad \text{or} \quad \sum P$$

and the theorem is established.

Remark. If we delete the subscript letters throughout in formulas (3.5) and (3.7), they become (3.1) and (3.2), respectively. Consequently, equations (4.3) and (4.4) with the necessary deletions constitute the proof of the purely contravariant contraction theorem which stated briefly is:

THEOREM 4.3. *The* M*th intrinsic derivative of the tensor* T^{ab} *is given by* $E^{\alpha a \cdot \beta b} L^{c}_{\alpha a} L^{n}_{\beta b}$.

BIBLIOGRAPHY

1. H. V. Craig, *On tensors relative to the extended point transformation*, Amer. J. Math. vol. 59 (1937) pp. 764–774.

2. ———, *Vector and tensor analysis*, New York, 1943.

3. ———, *On the structure of certain tensors*, Mathematics Magazine vol. 1 (1947).

4. A. Kawaguchi, *Die Differentialgeometrie höherer Ordnung* I. *Erweiterte Koordinatentransformationen und Extensoren*, Journal of the Faculty of Science, Hokkaiddo Imperial University (1) vol. 9 (1940).

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