SEQUENCES OF IDEAL SOLUTIONS IN THE TARRY-ESCOTT PROBLEM

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1. Introduction. In the Tarry-Escott problem (sometimes called the problem of multi-degree equalities, or of equal sums of like powers), one seeks integral solutions of the k equations

$$(1)\qquad \sum_{i=1}^{s} x_i^l = \sum_{i=1}^{s} y_i^l, \qquad l = 1, 2, \cdots, k.$$

The usual notation is to represent a solution of (1) by

$$(2)\qquad a_1, \cdots, a_s \overset{k}{=} b_1, \cdots, b_s.$$

Such a solution is called *trivial* if the a's form a permutation of the b's, and will be called *semi-trivial* if any $a_i = a_j$ or $b_i = b_j$. A solution is said to be in *reduced* form when $\sum a_i = 0$, $(a_i, b_i) = 1$, and solutions having the same reduced form are called *equivalent* solutions.

It is easily shown that for nontrivial solutions, $s > k$. The case $s = k+1$, called the *ideal* or *optimum* case [1, 2],[1] is of particular interest in many applications [5]. For a given k, $N(k)$ is defined as the least value of s for which (1) has nontrivial solutions. It is known in general [7] that $N(k) \leqq k(k+1)/2$, but numerical examples [6] give $N(k) = k+1$ for $k = 1, 2, \cdots, 9$.

In order to decrease the number of the equations (1), many writers have imposed the conditions

$$(3)\qquad x_i = -y_i, \qquad i = 1, 2, \cdots, s, \text{ for } s \text{ odd},$$

or

$$(4)\qquad x_{s+1-i} = -x_i, \quad y_{s+1-i} = -y_i, \qquad i = 1, 2, \cdots, s/2, \text{ for } s \text{ even}.$$

Solutions of (1) subject to (3) or (4) will be called *symmetric* solutions. It is evident that the conditions for symmetry are sufficient to assure that symmetric solutions are reduced.

By use of the binomial theorem, one finds that (2) implies

$$(5)\qquad Ma_1 + K, \cdots, Ma_s + K \overset{k}{=} Mb_1 + K, \cdots, Mb_s + K,$$

and

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[1] Numbers in brackets refer to the references cited at the end of the paper.

$$(6)\qquad a_1, \cdots, a_s, b_1 + h, \cdots, b_s + h \overset{k+1}{=} b_1, \cdots, b_s, a_1 + h, \cdots, a_s + h,$$

where M, K and h are any integers.

Equation (6) shows the existence of multi-degree equalities of arbitrarily high degree, but in general each time the degree is raised by one, the number of terms on each side of the equality is doubled. However, by taking for h a difference $|a_i - a_j|$ or $|b_i - b_j|$ which occurs t times, t terms on one side will cancel with the same number of terms on the other side. The set of all differences $|a_i - a_j|$ and $|b_i - b_j|$ will be called the *difference table* for a given equality.

If the process indicated in (6) is applied to a nontrivial ideal solution and leads to another nontrivial ideal solution, these ideal solutions will be said to be *in sequence*. Thus for a nontrivial ideal equality of degree k to be in sequence with one of degree $k+1$, the value used for h in (6) must occur exactly $2(k+1) - (k+2) = k$ times in the difference table for the ideal equality of degree k.

The following sequence of ideal solutions has been noted [5].

$$
\begin{array}{lrcl}
 & 1, 9 & \overset{1}{=} & 4, 6, \\
h = 2, & 1, 8, 9 & \overset{2}{=} & 3, 4, 11, \\
(7)\quad h = 1, & 1, 5, 8, 12 & \overset{3}{=} & 2, 3, 10, 11, \\
h = 7, & 1, 5, 9, 17, 18 & \overset{4}{=} & 2, 3, 11, 15, 19, \\
h = 8, & 1, 5, 10, 18, 23, 27 & \overset{5}{=} & 2, 3, 13, 15, 25, 26.
\end{array}
$$

It is the purpose of this paper to determine all sequences of ideal solutions which begin with degrees one, two and three (the nonsymmetric case for degree three in not completed). The appearance of certain substitution groups in the solutions, apparently not previously observed, will be noted. These groups are then used to simplify many of the computations which would otherwise be very tedious. Although the methods of the paper do not lead to new solutions, it is believed that the tabulation of complete information concerning sequences beginning with low degrees sheds new light on the structure of existing solutions.

2. Sequences beginning with the first degree. All reduced ideal solutions of first degree are of necessity in the symmetric form

$$a, -a \stackrel{1}{=} b, -b. \tag{8}$$

If we apply (6) with $h=2a$ ($h=2b$ gives similar results) and then (5) with $M=1$ and $K=-a$ to (8), there results

$$a+b, a-b, -2a \stackrel{2}{=} -a-b, -a+b, 2a. \tag{9}$$

The difference table for this equality contains $3a+b$, $3a-b$ and $2b$ each repeated two times. Applying (6) to (9) with $h=3a+b$, followed by (5) with $M=2$ and $K=-3a-b$, we have

$$\pm(7a+b, a+3b) \stackrel{3}{=} \pm(5a+3b, 5a-b). \tag{10}$$

The relationship of (10) and other solutions found in this and succeeding sections to results given in [**2**] will be pointed out later.

When the difference table for (10) is formed, it is found by equating all a_i-a_j with each other and with all b_i-b_j that for the same difference to occur exactly three times (and thus lead to another nontrivial ideal solution), a/b must have one of the following sixteen values: 3, 1/2, −5, 1/4, −1/4, −3/7, −1/6, −5/9, −3/11, −2/5, 1/7, −2, 2, 3/5, −2/3, −1/9. Fourteen distinct ideal equalities of degree 1 are obtained from these, leading to only six distinct ideal equalities of degree 2, which in turn give rise to four ideal equalities of degree 3. These imply two ideal equalities of degree 4 and finally three ideal equalities of degree 5. Here the sequences end. The relationship of these qualities to each other is shown in the accompanying chart (p. 390).

The ideal equality

$$\pm(1, 11, 12) \stackrel{5}{=} \pm(4, 9, 13)$$

is the reduced form of the fifth degree equality in (7) and is of particular interest. Two applications of (6) to the last equality of (7) give

$$h = 13, \quad 1, 5, 10, 16, 27, 28, 38, 39 \stackrel{6}{=} 2, 3, 13, 14, 25, 31, 36, 40,$$

$$h = 11, \quad 1, 5, 10, 24, 28, 42, 47, 51 \stackrel{7}{=} 2, 3, 12, 21, 31, 40, 49, 50.$$

Thus we have an example of an equality of degree 6 which, although not itself ideal, leads to an ideal equality of degree 7 (see [**2**, p. 632 (C)]). This particular ideal equality of degree 7 was first found by Tarry and has been used as the basis of extensive calculations [**8**]. Until recently [**6**], it was not known if ideal equalities of degree greater than 7 existed. The best results for $N(k)$, $k>9$, are still given by numerical examples rather than by theoretical methods.

Application of (6) to (9) with $h=3a-b$ followed by (5) with $M=2$ and $K=-3a+b$ leads to

$$\pm(7a-b,\ a-3b) \overset{3}{=} \pm(5a-3b,\ 5a+b), \tag{11}$$

and with $h=2b$ followed by (5) with $M=1$ and $K=-b$ leads to

$$\pm(2a+b,\ a-2b) \overset{3}{=} \pm(2a-b,\ a+2b). \tag{12}$$

However, neither (11) nor (12) leads to further ideal solutions differing from those in the chart.

We now summarize the results of this section in the following theorem.

THEOREM 1. *Every reduced ideal equality of degree* 1 *is symmetric and implies an ideal equality of degree* 2 *of the form* (9), *which in turn implies three ideal equalities of degree* 3 *of the form* (10), (11) *and* (12). *The only sequences beginning with degree* 1 *which extend to degrees* 4 *and* 5 *are equivalent to those of the chart*; *none extends beyond degree* 5.

3. **Sequences beginning with the second degree.** The general ideal solution of second degree [**3**, p. 52] is

$$AG+BD,\quad AD,\quad BG \overset{2}{=} AD+BG,\quad AG,\quad BD, \tag{13}$$

where A, B, D, G are any integers. Let $A=\lambda B$, $G=\mu D$, and (13) becomes

$$\lambda\mu+1,\quad \lambda,\quad \mu \overset{2}{=} \lambda+\mu,\quad \lambda\mu,\quad 1, \tag{14}$$

where λ and μ are rational numbers. When (14) is put in reduced form by the application of (5) with $M=3$ and $K=-\lambda\mu-\lambda-\mu-1$, it is found that the conditions for trivial solutions are λ or $\mu=0,\ 1$, and for symmetric solutions are λ or $\mu=-1,\ 2,\ 1/2$.

The symmetric ideal solutions of degree two are all equivalent to

$$\lambda,\quad 1-\lambda,\quad -1 \overset{2}{=} -\lambda,\quad \lambda-1,\quad 1, \tag{15}$$

given directly by (14) with $\mu=-1$, with $\mu=2$ followed by (5) with $M=1$ and $K=-\lambda-1$, or with $\mu=1/2$ followed by (5) with $M=2$ and $K=-\lambda-1$. If $\lambda=(a+b)/2a$, (15) reduces to (9), which is therefore equivalent to the general symmetric solution of degree two. Also, we now see that there are no symmetric sequences which *begin* with degree two.

The fact that special cases of (14) are given by λ or $\mu=0$, 1 and -1, 2, 1/2 suggests a possible connection with the cross-ratio group G_6 of projective geometry. It is easy to show that (15) is invariant under the substitutions λ, $1-\lambda$, $(\lambda-1)/\lambda$, $1/\lambda$, $1/(1-\lambda)$, $\lambda/(\lambda-1)$ which form this group.

The difference table for (15) contains $\lambda+1$, $\lambda-2$, $2\lambda+1$ each twice, and the equality in λ which corresponds to (10) is

$$(10') \qquad \pm (3\lambda + 1, \lambda - 3) \overset{3}{=} \pm (3\lambda - 1, \lambda + 3).$$

It is evident that (10′) is invariant under the substitutions of the group λ, $-\lambda$, $1/\lambda$, $-1/\lambda$. The values of λ for which (10′) leads to an ideal equality of degree 4 (corresponding to the sixteen values of a/b given in the preceding section) are $\pm 2/3$, $\pm 3/2$, $\pm 2/5$, $\pm 5/2$, $\pm 3/4$, $\pm 4/3$, ± 4, $\pm 1/4$. Hence we now see why these sixteen values give only four distinct equalities of degree 3.

Corresponding to (11) and (12) are

$$(11') \qquad \pm (3\lambda - 4, \lambda + 2) \overset{3}{=} \pm (3\lambda - 2, \lambda - 4)$$

which is invariant under λ, $2-\lambda$, $\lambda/(\lambda-1)$, $(\lambda-2)/(\lambda-1)$, and

$$(12') \qquad \pm (4\lambda - 1, 2\lambda - 3) \overset{3}{=} \pm (4\lambda - 3, 2\lambda + 1),$$

invariant under λ, $1-\lambda$, $\lambda/(2\lambda-1)$, $(\lambda-1)/(2\lambda-1)$. The groups for (10′), (11′) and (12′) are abstractly the same G_4.

If neither λ nor $\mu=0$, ± 1, 1/2, 2 in (14) we have a nontrivial, nonsymmetric solution. Forming the difference table, we find that for (14) to give rise to an ideal equality of degree 3,

$$\begin{aligned}\mu = \lambda, \quad & 1 - \lambda, \quad \lambda/(\lambda - 1), \quad 1/\lambda, \quad 1/(1 - \lambda), \quad (\lambda - 1)/\lambda,\\ & (\lambda + 1)/(2 - \lambda), \quad (\lambda - 2)/(2\lambda - 1), \quad (2\lambda - 1)/(\lambda + 1),\\ & (\lambda + 1)/(2\lambda - 1), \quad (1 - 2\lambda)/(2 - \lambda), \quad (2 - \lambda)/(1 + \lambda).\end{aligned}$$

These substitutions form a G_{12}. The first six are, of course, the cross ratio subgroup, and lead to semi-trivial solutions. The remaining six all give

$$(16) \qquad \lambda^2 - 2\lambda,\ -\lambda^2 + 1,\ 2\lambda - 1 \overset{2}{=} 0,\ \lambda^2 - \lambda + 1,\ -\lambda^2 + \lambda - 1,$$

which is invariant under the G_{12}. An examination of the difference table for (16) shows that a nontrivial equality of degree 3 can be formed from (16) by the application of (6) only if $h=\lambda^2-\lambda+1$. Making this application followed by (5) with $M=2$ and $K=-\lambda^2+\lambda-1$, we have

$$(17)\quad \begin{aligned} &3\lambda^2-3\lambda+3,\ -\lambda^2+5\lambda-3,\ -3\lambda^2+\lambda+1,\ \lambda^2-3\lambda-1 \\ &\overset{3}{=}\lambda^2+3\lambda-1,\ -\lambda^2-\lambda+3,\ 3\lambda^2-5\lambda+1,\ -3\lambda^2+3\lambda-3, \end{aligned}$$

which is also invariant under the G_{12}.

The difference table for (17) is found to contain the same value three times (and one which will lead to a nontrivial equality of degree 4) for twelve numerical values of λ. One of these is 3 and the others can be obtained by substituting $\lambda=3$ in the last eleven elements of the G_{12}. The only sequence given is the following.

$$(18)\quad \begin{aligned} & & 7,\ 0,\ -7 \overset{2}{=} 5,\ 3,\ -8, \\ &h=7, & 21,\ 3,\ -1,\ -23 \overset{3}{=} 17,\ 13,\ -9,\ -21, \\ &h=22, & 14,\ 12,\ -4,\ -5,\ -17 \overset{4}{=} 16,\ 7,\ 3,\ -10,\ -16. \end{aligned}$$

Replacing λ by c/d in (16) and (17), we have the following theorem.

THEOREM 2. *The only sequences of ideal solutions beginning with degree two are nonsymmetric and are equivalent to*

$$c^2-2cd,\ -c^2+d^2,\ 2cd-d^2 \overset{2}{=} 0,\ c^2-cd+d^2,\ -c^2+cd-d^2,$$
$$3c^2-3cd+3d^2,\ -c^2+5cd-3d^2,\ -3c^2+cd+d^2,$$
$$c^2-3cd-d^2 \overset{3}{=} c^2+3cd-d^2,$$
$$-c^2-cd+3d^2,\ 3c^2-5cd+d^2,\ -3c^2+3cd-3d^2,$$

or to (18).

4. **Sequences beginning with the third degree.** A. *Symmetric case.* The general symmetric solution of third degree is [**2**, (4)]

$$(19)\qquad a_1,\ a_2,\ -a_1,\ -a_2 \overset{3}{=} b_1,\ b_2,\ -b_1,\ -b_2$$

where $a_1=p_1p_2+p_3p_4$, $b_1=p_1p_2-p_3p_4$, $a_2=p_1p_3-p_2p_4$, $b_2=p_1p_3+p_2p_4$. Solution (10′) is the special case $p_1=\lambda$, $p_2=3$, $p_3=p_4=1$, (11′) the case $p_1=2$, $p_2=\lambda$, $p_3=\lambda-2$, $p_4=-1$, and (12′) the case $p_1=2\lambda-1$, $p_2=1$, $p_3=2$, $p_4=-1$.

Let $p_2=\alpha p_3$, $p_4=\beta p_1$. Then (19) becomes

$$(20)\qquad \pm(\alpha+\beta,\ 1-\alpha\beta) \overset{3}{=} \pm(\alpha-\beta,\ 1+\alpha\beta)$$

where α and β are rational, and where α and β play an interchangeable role. Equality (20) is trivial if α or $\beta=0, \pm 1$, and semi-trivial if

$$\beta = \alpha,\ -\alpha,\ 1/\alpha,\ -1/\alpha,\ (1+\alpha)/(1-\alpha),\ -(1+\alpha)/(1-\alpha),$$
$$(1-\alpha)/(1+\alpha),\ -(1-\alpha)/(1+\alpha).$$

These substitutions form a G_8. Three equal differences leading to a nontrivial equality of degree 4 occur in the difference table for (20) if $\beta, -\beta, 1/\beta, -1/\beta=(3\alpha+1)/(\alpha+1), (3\alpha-1)/(\alpha-1), (\alpha+3)/(\alpha+1), (\alpha-3)/(\alpha-1), (\alpha+3)/(3\alpha-1), (3\alpha+1)/(3-\alpha)$. But since (20) is invariant under $\beta, -\beta, 1/\beta, -1/\beta$ we need consider only the six expressions in α. Finally, since the first four expressions are invariant under $\alpha, -\alpha, 1/\alpha, -1/\alpha$, and the last two under α and $1/\alpha$, we need consider only $\beta=(3\alpha+1)/(\alpha+1)$ and $\beta=(\alpha+3)/(3\alpha-1)$.

(i) Let $\beta=(3\alpha+1)/(\alpha+1)$. Then (20) becomes

$$(21)\quad \pm (3\alpha^2+2\alpha+1,\ \alpha^2-2\alpha-1) \overset{3}{=} \pm (3\alpha^2-1,\ \alpha^2+4\alpha+1),$$

which is trivial if $\alpha=0, \pm 1, -1/2, -1/3$ and which is given by [**2**, (11)] with $a=\alpha^2+4\alpha+1$, $b=3\alpha^2+2\alpha+1$, and $k=\alpha^2-2\alpha-1$. The only value for h leading to a nontrivial ideal equality of degree 4 is $2\alpha^2-4\alpha-2$, giving

$$(22)\quad \begin{aligned} &2\alpha^2-\alpha-1,\ \alpha^2+2\alpha+1,\ -3\alpha-1,\ -\alpha^2+2\alpha+1,\ -2\alpha^2 \\ &\overset{4}{=} 2\alpha^2,\ \alpha^2-2\alpha-1,\ 3\alpha+1,\ -\alpha^2-2\alpha-1,\ -2\alpha^2+\alpha+1. \end{aligned}$$

There are sixteen values of α for which the sequence continues, twelve of them (in three sets of four each) when substituted in (21) leading to

$$\pm (1, 17) \overset{3}{=} \pm (11, 13), \qquad \pm (3, 11) \overset{3}{=} \pm (7, 9),$$
$$\pm (1, 13) \overset{3}{=} \pm (7, 11),$$

all of which appear in the chart. The remaining four values, $-3/2$, $1/5$, $-1/7$, $-3/4$ all give

$$\pm (11, 23) \overset{3}{=} \pm (17, 19),$$

$$(23)\quad h = 34, \quad 18, 17, -1, -14, -20 \overset{4}{=} 20, 14, 1, -17, -18,$$

$$h = 19, \qquad \pm (15, 47, 59) \overset{5}{=} \pm (9, 53, 55).$$

(ii) Let $\beta=(\alpha+3)/(3\alpha-1)$. Then (20) becomes

$$(24)\quad \pm (3\alpha^2+3,\ \alpha^2+1) \overset{3}{=} \pm (3\alpha^2-2\alpha-3,\ \alpha^2+6\alpha-1),$$

which is trivial if $\alpha=0,\ \pm 1,\ 2,\ -1/2,\ -3,\ 1/3$, and which is given by [2, (13)] with $u=3\alpha^2-2\alpha-3,\ v=\alpha^2+6\alpha-1,\ k=\alpha^2+1$. For $\alpha=-2$, (24) gives the fourth equality of degree 3 of the chart, namely

$$\pm(5, 15) \overset{3}{=} \pm(9, 13).$$

The sequence beginning with (24) continues for $h=\alpha^2+1$ with

$$(25)\quad \begin{aligned} &2\alpha^2-\alpha-1,\ \alpha^2+3\alpha,\ -3\alpha+1,\ -\alpha^2+\alpha+2,\ -2\alpha^2-2 \\ &\overset{4}{=} 2\alpha^2+2,\ \alpha^2-\alpha-2,\ 3\alpha-1,\ -\alpha^2-3\alpha,\ -2\alpha^2+\alpha+1, \end{aligned}$$

and then for $h=\alpha^2-4\alpha-1$ with

$$(26)\quad \begin{aligned} &\pm(5\alpha^2-4\alpha+3,\ 3\alpha^2-6\alpha-5,\ \alpha^2+10\alpha+1) \\ &\overset{5}{=} \pm(5\alpha^2-6\alpha-3,\ 3\alpha^2+4\alpha+5,\ \alpha^2-10\alpha+1), \end{aligned}$$

or for $h=2\alpha^2+2\alpha-2$ with

$$(27)\quad \begin{aligned} &\pm(3\alpha^2+\alpha+1,\ 2\alpha^2-3,\ \alpha^2+4\alpha-2) \\ &\overset{5}{=} \pm(3\alpha^2-2,\ 2\alpha^2+4\alpha-1,\ \alpha^2-\alpha+3). \end{aligned}$$

However, there are no real values of α giving an ideal equality of degree 6.

B. *Nonsymmetric case.* The general nonsymmetric solution of third degree is [2, (10)]

$$(28)\quad \begin{aligned} a_1 &= -mr+ns+t, & b_1 &= -r+s+mnt, \\ a_2 &= mr-ns+t, & b_2 &= r-s+mnt, \\ a_3 &= mr+ns-t, & b_3 &= r+s-mnt, \\ a_4 &= -mr-ns-t, & b_4 &= -r-s-mnt, \end{aligned}$$

where m and n are rational numbers not equal to 0, ± 1 and where

$$\begin{aligned} r &= -(m^2-1)nU^2-2(n^2-1)UV+(n^2-1)nV^2, \\ s &= (m^2-1)U^2-2(m^2-1)nUV-(n^2-1)V^2, \\ t &= (m^2-1)U^2+(n^2-1)V^2, \end{aligned}$$

with U and V relatively prime integers.

Equality (17) is given by (28) with $m=\lambda,\ n=-(\lambda-1)/\lambda$, $t=-(\lambda-2),\ r=\lambda+1,\ s=\lambda(2\lambda-1)$. The equality of degree 3 of (18) is given by $m=5,\ n=3,\ U=1,\ V=-3$.

To determine sequences starting with (28), it is best to form the difference table for the transformation used in [**2**, (6)]. This leads to a series of linear conditions which are then incorporated in [**2**, (8)] and finally in the parametric solution. The following example shows the existence of such sequences. With $m=n=2$, $U=3$, $V=1$, we have

$$31, 13, -21, -23 \overset{3}{=} 29, 11, -7, -33,$$

$$h = 18, \quad 47, 13, -15, -21, -23 \overset{4}{=} 49, -3, -5, -7, -33.$$

THEOREM 3. *All symmetric sequences beginning with degree three are equivalent to* (21)(22), *to* (24)(25)(26) *or* (27) (*in all of which* α *is rational*), *or to* (23). *Nonsymmetric sequences beginning with degree three exist and can be determined by the method of this paper.*

5. **Sequences beginning with higher than the third degree.** General parametric ideal solutions have not been found beyond the third degree, but many special solutions are known [**1**, **2**, **4**, **6**]. Although probably only sequences beginning with low degrees extend for more than 2 or 3 steps, sequences beginning with high degrees do exist. This paper will be concluded with two numerical examples of sequences which were obtained from known ideal equalities of degrees 6 and 8 by the use of a theorem of Escott [**5**, page 617] which gives the *predecessor* of a given equality. In neither case can the sequence be extended in either direction.

The first example is

$$1, 18, 35, 59, 72 \overset{4}{=} 2, 15, 39, 56, 73,$$

$$h = 17, \quad 1, 19, 32, 59, 72, 90 \overset{5}{=} 2, 15, 39, 52, 76, 89,$$

$$h = 13, \quad 1, 19, 28, 59, 65, 90, 102 \overset{6}{=} 2, 14, 39, 45, 76, 85, 103,$$

where the equality of degree 6 is due to Escott [**5**, p. 616].

The second example is

$$1, 25, 31, 72, 87, 128, 134, 158 \overset{7}{=} 2, 18, 43, 59, 100, 116, 141, 157,$$

$$h = 41, \quad 1, 25, 31, 84, 87, 134, 158, 182, 198 \overset{8}{=} 2, 18, 42, 66, 113, 116, 169, 175, 199,$$

where the equality of degree 8 is due to Letac [**6**, p. 48].

$\pm 1 \overset{1}{=} \pm 3 \quad \pm 1 \overset{1}{=} \pm 2 \quad \pm 1 \overset{1}{=} \pm 5 \quad \pm 1 \overset{1}{=} \pm 4 \quad \pm 3 \overset{1}{=} \pm 7 \quad \pm 1 \overset{1}{=} \pm 6 \quad \pm 5 \overset{1}{=} \pm 9 \qquad \pm 3 \overset{1}{=} \pm 11 \quad \pm 2 \overset{1}{=} \pm 5 \quad \pm 1 \overset{1}{=} \pm 7 \quad \pm 1 \overset{1}{=} \pm 2 \quad \pm 3 \overset{1}{=} \pm 5 \quad \pm 2 \overset{1}{=} \pm 3 \quad \pm 1 \overset{1}{=} \pm 9$

$h=6 \quad h=2 \quad h=2 \quad h=10 \quad h=2 \quad h=6 \quad h=14 \quad h=2 \quad h=10 \qquad h=6 \quad h=4 \quad h=14 \quad h=2 \quad h=4 \quad h=6 \quad h=10 \quad h=4 \quad h=2$

$3, -1, -2 \overset{2}{=} 2, 1, -3 \quad 3, 2, -5 \overset{2}{=} 5, -2, -3 \quad 5, 2, -7 \overset{2}{=} 7, -2, -5 \qquad 4, 3, -7 \overset{2}{=} 7, -3, -4 \quad 3, 1, -4 \overset{2}{=} 4, -1, -3 \quad 4, 1, -5 \overset{2}{=} 5, -1, -4$

$h=5 \quad h=1 \quad h=7 \quad h=3 \qquad h=1 \quad h=7 \quad h=5 \quad h=3$

$\pm(3, 11) \overset{3}{=} \pm(7, 9) \quad \pm(1, 17) \overset{3}{=} \pm(11, 13) \qquad \pm(5, 15) \overset{3}{=} \pm(9, 13) \quad \pm(1, 13) \overset{3}{=} \pm(7, 11)$

$h=14 \quad h=2 \qquad h=10 \quad h=14$

$8, 7, -1, -5, -9 \overset{4}{=} 9, 5, 1, -7, -8 \qquad 9, 7, -2\ -4, -10 \overset{4}{=} 10, 4, 2, -7, -9$

$h=8 \qquad h=4 \quad h=2 \qquad h=11$

$\pm(1, 11, 12) \overset{5}{=} \pm(4, 9, 13) \qquad \pm(1, 9, 10) \overset{5}{=} \pm(5, 6, 11) \qquad \pm(9, 25, 29) \overset{5}{=} \pm(15, 19, 31)$

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