MEAN VALUES OF PERIODIC FUNCTIONS

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Let $L^p$ denote the class of complex measurable functions of period $2\pi$ for which $M_p(f) < \infty$, where

\[
M_p(f) = \left( \int_0^{2\pi} |f(x)|^p \, dx \right)^{1/p} \quad (1 \leq p < \infty),
\]

(1)

Let $M_\infty(f) = \text{ess. sup}_{0 \leq x \leq 2\pi} |f(x)|$.

(2)

Let $K_{m,p}$ denote the subset of $L^p$ whose elements, $f(x)$, have a Fourier series of the form

\[
\sum_{n=\pm m}^\infty (a_n \cos nx + b_n \sin nx) \quad (m \geq 1).
\]

(3)

The functions of $K_{m,p}$ and their Fourier series (3) are transformed by the real number $\delta$ and the sequence of real numbers $\lambda = \{\lambda(n)\}$ into the series

\[
\sum_{n=\pm m}^\infty \lambda(n) \left\{ a_n \cos \left( nx + \frac{\delta \pi}{2} \right) + b_n \sin \left( nx + \frac{\delta \pi}{2} \right) \right\}
\]

(4)

\[
= \sum_{n=\pm m}^\infty \lambda(n) \left\{ \left( a_n \frac{\delta \pi}{2} + b_n \frac{\delta \pi}{2} \right) \cos nx + \left( b_n \frac{\delta \pi}{2} - a_n \frac{\delta \pi}{2} \right) \sin nx \right\}.
\]

A slight modification of the well known result\(^1\) [5, pp. 100 ff.]\(^2\) for the case in which $\delta = 0$ shows that if

\[
\sum_{n=\pm m}^\infty \lambda(n) \cos \left( nx - \frac{\delta \pi}{2} \right) = \sum_{n=\pm m}^\infty \lambda(n) \left\{ \cos \frac{\delta \pi}{2} \cos nx + \sin \frac{\delta \pi}{2} \sin nx \right\}
\]

(5)

Presented to the Society, November 30, 1946; received by the editors December 16, 1946.

\(^1\) Although the convention is adopted in Trigonometrical series that $f(x)$ is real, the results of the sections of Trigonometrical series to which reference is made in this note hold for complex $f(x)$.

\(^2\) Numbers in brackets refer to the bibliography at the end of the paper.

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is a Fourier or even a Fourier-Stieltjes series then (4) is the Fourier series of a function of $L^p$, $1 \leq p \leq \infty$. Throughout the sequel it is assumed that (5) is a Fourier series. Series (4) is therefore the Fourier series of a function $g(x) \in L^p$ and is of the form of (3), hence $g(x) \in K_{m,p}$. The transformation determined by the number $\delta$ and the sequence $\lambda$ is thus a transformation of $K_{m,p}$ onto itself.

The objective of the present note is to establish an inequality between the means $M_p(f)$ and $M_p(g)$ which holds for all $f(x) \in K_{m,p}$. For the essentially bounded case this has been done by B. v. Sz. Nagy [3], and for completeness the result is stated as a lemma.

**Lemma 1 [Sz. Nagy].** If $f(x) \in K_{m,\infty}$, then (4) is the Fourier series of a continuous function $g(x) \in K_{m,\infty}$ and

$$M_\infty(g) \leq A(\lambda, \delta, m) M_\infty(f),$$

where $A(\lambda, \delta, m)$ is a function only of the indicated variables and not of the particular $f(x) \in K_{m,\infty}$.

The notation $A(\lambda, \delta, m)$ will be used throughout the sequel to denote the smallest possible function which will satisfy (6) for all $f(x) \in K_{m,\infty}$.

**Lemma 2.** If $f(x) \in K_{m,2}$, then (4) is the Fourier series of a function $g(x) \in K_{m,2}$ and

$$M_2(g) \leq \Lambda(m) M_2(f),$$

where $\Lambda(m) = \max_{n \geq 0} |\lambda(n)|$.

The Riesz-Fischer theorem asserts that (4) is the Fourier series of a function $g(x) \in L^2$ and that

$$M_2(g) = \left\{ \pi \sum_{n=-\infty}^{\infty} (\lambda(n))^2 (|a_n|^2 + |b_n|^2) \right\}^{1/2} \leq \Lambda(m) \left\{ \pi \sum_{n=-\infty}^{\infty} (|a_n|^2 + |b_n|^2) \right\}^{1/2} = \Lambda(m) M_2(f).$$

If $\Lambda(m) = |\lambda(r)|$ ($r \geq m$), then $f(x) = \cos rx$ gives equality in (7).

It is now possible to state the principal theorem.

**Theorem 1.** If $f(x) \in K_{m,p}$, then (4) is the Fourier series of a function $g(x) \in K_{m,p}$ and

$$M_p(g) \leq \Lambda^{2/p}(m) A^{(p-2)/p}(\lambda, \delta, m) M_p(f) \quad (2 \leq p < \infty),$$

$$M_p(g) \leq B_p \Lambda^{2/p'}(m) A^{(p'-2)/p'}(\lambda, -\delta, m) M_p(f) \quad (1 < p \leq 2)$$

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where Δ(m) and A(λ, δ, m) are defined in Lemmas 1 and 2, B_p is a constant depending only on p and not on the particular f(x) ∈ K_{m,p}, and p' = p/(p - 1).

The transformation of f(x) into g(x), or of series (3) into series (4), is a linear transformation of K_{m,2} onto itself, and also of K_{m,∞} onto itself. The direct application of an interpolation scheme for L^p fails in the attempt to establish (8) since the space K_{m,p} is a nondense linear subspace of L^p. However, the proof of the interpolation result for L^p as given in Trigonometrical series [5, p. 198 ff.] carries through for the space K_{m,p} on the basis of the following lemma.

**Lemma 3.** The step functions of K_{m,p} are dense in K_{m,p} in the metric of L^p (1 < p < ∞).

The step functions of K_{m,p} are those functions of K_{m,p} which assume only a finite number of values and assume each of these values on a finite sum of intervals in (0, 2π). Suppose f(x) ∈ K_{m,p} and η is a positive number. The density of the continuous functions of L^p requires the existence of a continuous function h(x) such that M_p(f - h) < η and |c_0|/2 + \sum_{n=1}^{m-1} (|c_n| + |d_n|) < η. The function h(x) = h(x) - c_0/2 - \sum_{n=1}^{m-1} (c_n \cos nx + d_n \sin nx) is therefore a continuous function of class K_{m,p} and M_p(f - h) ≤ M_p(f - h) + η(2π) < η. Hence the continuous functions of K_{m,p} are dense in K_{m,p}. It is sufficient therefore to show that the continuous functions of L^p can be approximated uniformly by step functions of K_{m,p}.

Consider first a continuous f(x) ∈ K_{1,p}. For any positive η, there is a step function s(x) such that |f(x) - s(x)| ≤ η for all x. If c = (1/2π) \int_0^{2π} s(x) dx, then since \int_0^{2π} f(x) dx = 0, |c| ≤ (1/2π) \int_0^{2π} |s(x) - f(x)| dx ≤ η. The step function t(x) = s(x) - c is therefore in K_{1,p} and |f(x) - t(x)| ≤ 2η.

Suppose next that it has been demonstrated that the continuous functions of K_{r,p} can be uniformly approximated by step functions of K_{s,p} for 1 ≤ r < s. Since K_{r,p} ⊇ K_{s,p} if r < s, any continuous function of K_{m,p} can be uniformly approximated by step functions whose Fourier coefficients of order less than (m - 1) vanish. Hence if f(x) is a continuous function of K_{m,p} and η is a positive number, there is a step function s(x) ∈ K_{m-1,p} such that |f(x) - s(x)| < η for all x. Suppose that c = (1/4) \int_0^{2π} s(x) \cos (m - 1)x dx and d = (1/4) \int_0^{2π} s(x) \sin (m - 1)x dx. Since f(x) ∈ K_{m,p}, both |c| < 2η and |d| < 2η. Suppose the function t(x) = s(x) - c \text{ sgn} \cos (m - 1)x - d \text{ sgn} \sin (m - 1)x, where \text{ sgn} u = 0 if u = 0 and \text{ sgn} u = u/|u| if u ≠ 0. It can be shown by direct calculation that the step function t(x) ∈ K_{m,p}. Since |f(x) - t(x)| ≤ |f(x) - s(x)|
$+$ $c \sgn \cos (m-1)x$ $+$ $d \sgn \sin (m-1)x$ $< 5\eta$, the function $t(x)$ gives the desired uniform approximation.

In order to establish (9), it is first noted [5, p. 105] that since $g(x) \in L^p$,

(10) $$M_p(g) = \sup \left| \int_0^{2\pi} g(x) \overline{h(x)} dx \right|$$

with the supremum taken over all $h(x)$ for which $M_p(h) \leq 1$. Hence if $\eta$ is a positive number there is an $h(x)$ for which

$$M_p(h) \leq 1$$

and

(11) $$M_p(g) - \eta \leq \left| \int_0^{2\pi} g(x) \overline{h(x)} dx \right| .$$

Suppose that $h(x) \sim r_0/2 + \sum_{n=1}^{\infty} (r_n \cos nx + s_n \sin nx)$ and that $h_m(x) \sim \sum_{n=-m}^{\infty} (r_n \cos nx + s_n \sin nx)$. A double application of Parseval’s relation for functions of $L^p$ and $L^{p'}$ shows that

(12) $$\int_0^{2\pi} g(x) \overline{h(x)} dx = \pi \sum_{n=m}^{\infty} \left\{ \lambda(n) \left( a_n \cos \frac{\delta\pi}{2} + b_n \sin \frac{\delta\pi}{2} \right) \frac{r_n}{s_n} + \lambda(n) \left( b_n \cos \frac{\delta\pi}{2} - a_n \sin \frac{\delta\pi}{2} \right) s_n \right\}$$

$$= \pi \sum_{n=m}^{\infty} \left\{ \lambda(n) \left( \frac{r_n \cos \frac{\delta\pi}{2} - s_n \sin \frac{\delta\pi}{2}}{a_n} \right) \right\}$$

where

$$H(x) \sim \sum_{n=m}^{\infty} \lambda(n) \left\{ \left( r_n \cos \frac{\delta\pi}{2} - s_n \sin \frac{\delta\pi}{2} \right) \cos nx \right.$$

$$+ \left. \left( s_n \cos \frac{\delta\pi}{2} + r_n \sin \frac{\delta\pi}{2} \right) \sin nx \right\}$$

$$= \sum_{n=m}^{\infty} \lambda(n) \left\{ r_n \cos \left( nx - \frac{\delta\pi}{2} \right) + s_n \sin \left( nx - \frac{\delta\pi}{2} \right) \right\} .$$
Thus \( H(x) \) is the transform of \( h_m(x) \) which is obtained by use of the number \( -\delta \) and the sequence \( \lambda \). The application of Hölder's inequality followed by the use of (8) then shows that

\[
\int_0^{2\pi} \overline{H(x)} f(x) \, dx \leq M_{p'}(H) M_p(f) \\
\leq M_p(f) \Lambda^{2/p'}(m) A^{(p'-2)/p'}(\lambda, -\delta, m) M_{p'}(h_m).
\]

A well known result of M. Riesz [1] implies that

\[
M_{p'}(h_m) \leq B_p M_{p'}(h)
\]

where \( B_p \) depends only on \( p \) and not on the functions involved. The combination of formulas (10) through (14) then shows that

\[
M_{p}(g) - \eta \leq B_p \Lambda^{2/p'}(m) A^{(p'-2)/p'}(\lambda, -\delta, m) M_{p}(f)
\]

and (9) follows since \( \eta \) was arbitrary.

The result of Theorem 1 will now be applied to integrals of functions of \( K_{m,p} \). It is convenient to use the definition of the integral of order \( \alpha \) which is due to Weyl [4]. For any positive \( \alpha \), for \( f(x) \in K_{m,p} \) and with Fourier series (3), the integral of order \( \alpha \), \( f_\alpha(x) \), is defined as

\[
f_\alpha(x) = \sum_{n=-\infty}^{\infty} \frac{1}{n^\alpha} \left\{ a_n \cos \left( nx - \frac{\alpha \pi}{2} \right) + b_n \sin \left( nx - \frac{\alpha \pi}{2} \right) \right\}.
\]

Thus \( f_\alpha(x) \) is the transform of \( f(x) \) of the type of (4) with \( \delta = -\alpha \) and the sequence \( \{\lambda(n)\} = \{n^{-\alpha}\} \). Various results are known concerning the relationship between \( M_\alpha(f_\alpha) \) and \( M_\alpha(f) \), the most inclusive of which is that of Sz. Nagy [3] who shows that

\[
A(\lambda, \delta, m) = A(\{n^{-\alpha}\}, -\alpha, m)
\leq (4/\pi m^\alpha) \left\{ \cos \frac{\alpha \pi}{2} \left[ \sum_{v=0}^{\infty} (-1)^v (2v + 1)^{-1+\alpha} \right] + \sin \frac{\alpha \pi}{2} \left[ \sum_{v=0}^{\infty} (2v + 1)^{-1+\alpha} \right] \right\}
\leq (4/\pi m^\alpha) H(\alpha).
\]

It can also be seen from [3] that

\[
A(\{n^{-\alpha}\}, \alpha, n) \leq (4/\pi m^\alpha) H(\alpha).
\]

A direct application of Theorem 1 yields the following theorem.

**Theorem 2.** If \( f(x) \in K_{m.p} \), and \( f_\alpha(x) \) is its integral of order \( \alpha \) (\( \alpha \) not necessarily integral) then
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\[ M_p(f_a) \leq m^{-\alpha}(4H(\alpha)/\pi)^{(p^2 - 1)/p} M_p(f) \quad (2 \leq p), \]
\[ M_p(f_a) \leq B_p m^{-\alpha}(4H(\alpha)/\pi)^{(2-p)/p} M_p(f) \quad (1 < p \leq 2) \]

where \( B_p \) is the constant of Theorem 1 and

\[
H(\alpha) = \left| \cos \frac{\alpha \pi}{2} \sum_{n=0}^{\infty} (-1)^n (2n + 1)^{-(1+\alpha)} \right| + \left| \sin \frac{\alpha \pi}{2} \sum_{n=0}^{\infty} (2n + 1)^{-(1+\alpha)} \right|.
\]

A result of Schmidt [2] shows that for the real functions of \( K_1, p \) and \( \alpha \) integral the coefficient of \( M_p(f) \) in (15) is not the best possible.

BIBLIOGRAPHY


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