ON A CLASS OF TAYLOR SERIES

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1. Introduction. Consider the Taylor series \( \sum_{n=0}^{\infty} a_n z^n \). Suppose that the singularities of the function defined by the series all lie in certain regions of the complex plane and that the coefficients possess certain arithmetical properties. Mandelbrojt\(^1\) has shown that under restrictions of this nature it is possible to predict the form of the function defined by the series. This note is concerned with the establishing of a new method to obtain more general results of this nature.

2. The method. The method that is employed here is an adaptation of a method used by Lindelöf [2] in the problem of representation of a function defined by a series.

Let \( f(z) \) be regular in a region \( D \) of the complex plane. Suppose that there exists a linear transformation \( t = h(z) \) which maps the region of regularity into a region which includes the unit circle of the \( t \)-plane in its interior. Let \( z = g(t) \) be the inverse of this transformation. Then \( F(t) = f(g(t)) \) is regular in this region in the \( t \)-plane. For this note it is convenient to suppose that \( z = 0 \) corresponds to \( t = 0 \) in the mapping. We may expand \( g(t) \) in a Taylor series about \( t = 0 \) and obtain

\[
(2.1) \quad z = b_1 t + b_2 t^2 + \cdots
\]

convergent for \( t \) in absolute value sufficiently small. Let

\[
(2.2) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n
\]

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\(^1\) See, Mandelbrojt [3]. Numbers in brackets refer to the bibliography at the end of the paper.
be the element of $f(z)$ at the origin. For $t$ in absolute value sufficiently small we may substitute (2.1) in (2.2) and obtain

$$(2.3) \quad F(t) = f(g(t)) = \sum_{n=0}^{\infty} C_n t^n.$$  

We have seen, however, that $F(t) = f(g(t))$ is regular in a region in the $t$-plane which includes the unit circle in its interior. Hence the radius of convergence of (2.3) is greater than one. Therefore we may write

$$(2.4) \quad \limsup_{n \to \infty} (|C_n|)^{1/n} < 1.$$  

As the $C_n$ are polynomial combinations of the $a_n$ and $b_n$ we see that under certain circumstances (2.4) may imply $C_n = 0$ for $n$ greater than some $n_0$. For example, if the $C_n$ are all integers (2.4) implies the existence of an $n_0$ such that $C_n = 0$ for $n > n_0$. It is also clear that if $C_n = C'_n + iC''_n$ where $C'_n$ and $C''_n$ are integers that the conclusion $C_n = 0$, $n > n_0$, still holds. Under these circumstances we obtain upon substituting $t = h(z)$ in (2.3)

$$f(z) = \sum_{n=0}^{n_0} C_n [h(z)]^n.$$  

3. Applications. We now proceed to the proof of the principal theorem.

**Theorem 3.1.** If the series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has rational coefficients such that there exists an integer $L$ for which the quantities

$$a_0, a_1 L, a_2 L^2, \ldots, a_n L^n, \ldots$$

are integers, and if the function defined by the series is regular exterior to and on the circumference of a circle with center $(LK'/\left(K'^2 + K''^2 - 1\right), LK''/(K'^2 + K''^2 - 1))$ and radius $L/(K'^2 + K''^2 - 1)$ where $K'$ and $K''$ are integers, $K'^2 + K''^2 \neq 1$, then the function defined by the series is of the form

$$P(z)$$

$$(L/(K' + iK'') - z)^{n_0}$$

where $P(z)$ is a polynomial and $n_0$ is a positive integer.

It is easily shown that the transformation

$$(3.1) \quad t = \frac{z}{L - (K' + iK'')z}$$
maps the region of regularity into a region containing the unit circle of the \( t \)-plane in its interior. By substituting the solution of (3.1) for \( z \) in (2.2) and setting \( K = K' + iK'' \) we have

\[
F(t) = f(g(t)) = \sum_{n=0}^{\infty} a_n \frac{L^n t^n}{(1 + Kt)^n} = \sum_{n=0}^{\infty} a_n L^n t^n \sum_{m=0}^{\infty} C_{-n,m} (Kt)^m.
\]

Then since the series are absolutely convergent, we obtain

\[
F(t) = \sum_{n=0}^{\infty} \left[ a_n L^n C_{-n,0} + C_{-n,1} K L^{n-1} a_{n-1} + \cdots + a_0 C_{-n,n} K^n \right] t^n,
\]

(3.2) \( F(t) = \sum_{n=0}^{\infty} C_n t^n \).

Here \( C_n = C_n' + iC_n'' \) where \( C_n' \) and \( C_n'' \) are integers, for the \( C_{-n,m} \) are binomial coefficients and \( a_n L^n \) is an integer for all \( n \geq 0 \) by hypothesis. Also \( K' \) and \( K'' \) are integers. Hence, from the discussion in §2 it follows that

\[
\limsup_{n \to \infty} | C_n |^{1/n} < 1
\]

implies the existence of a number \( n_0 \) such that \( C_n = 0 \) for \( n > n_0 \). Therefore upon substituting (3.1) in (3.2) we have

\[
f(z) = \sum_{n=0}^{n_0} C_n \left( \frac{z}{L - Kz} \right)^n = \frac{P(z)}{(L/(K' + iK'') - z)^{n_0}}.
\]

This completes the proof of the theorem. If now we choose \( K'' = 0 \) and \( K' = L \) we have the theorem of Mandelbrojt [3]. This proof of Mandelbrojt's theorem has some points in common with a proof of the same theorem due to Achyser [1].\footnote{The author is indebted to the referee for this observation.} If in addition the \( a_n \) are all integers we may set \( L = 1 \) and have a new theorem.

We now proceed to the proof of the following theorem.

**Theorem 4.1.** If the series \( \sum_{n=0}^{\infty} a_n z^n \) has rational coefficients such that there exists an integer \( L \) for which the quantities

\[
a_0, a_1 L, a_2 L^2, \ldots, a_n L^n, \ldots
\]
are integers and if the function defined by the series is regular in the half-plane \( R(z) \leq L/2 \) including the point at infinity then the series defines a function of the form

\[
\frac{P(z)}{(L - z)^{n_0}}
\]

where \( P(z) \) is a polynomial and \( n_0 \) is a positive integer.

From the hypothesis it is easily seen that the transformation

\[
t = \frac{z}{L - z}
\]

maps the region of regularity into a region which includes the unit circle in the \( t \)-plane in its interior. Upon solving (3.3) for \( z \) and substituting in (2.2) we obtain

\[
F(t) = f \left( \frac{Lt}{1 + t} \right) = \sum_{n=0}^{\infty} a_n \left( \frac{Lt}{1 + t} \right)^n
\]

(3.4)

\[
= \sum_{n=0}^{\infty} a_n t^n L^n \sum_{m=0}^{\infty} C_{-n,m}(t)^m = \sum_{n=0}^{\infty} C_n t^n.
\]

Then by the same arguments employed in Theorem (3.1) it follows that there exists an \( n_0 \) such that \( C_n = 0 \) for \( n > n_0 \). Therefore by substituting from (3.3) in (3.4) we have

\[
f(z) = \sum_{n=0}^{n_0} C_n \left( \frac{z}{L - z} \right)^n = \frac{P(z)}{(L - z)^{n_0}},
\]

where \( P(z) \) is a polynomial and \( n_0 \) is a positive integer. This completes the proof of the theorem.

**Bibliography**