INNER PRODUCTS IN NORMED LINEAR SPACES

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Let $T$ be any normed linear space [1, p. 53]. Then an inner product is defined in $T$ if to each pair of elements $x$ and $y$ there is associated a real number $(x, y)$ in such a way that $(x, y) = (y, x)$, $\|x\|^2 = (x, x)$, $(x, y+z) = (x, y) + (x, z)$, and $(tx, y) = t(x, y)$ for all real numbers $t$ and elements $x$ and $y$. An inner product can be defined in $T$ if and only if any two-dimensional subspace is equivalent to Cartesian space [5]. A complete separable normed linear space which has an inner product and is not finite-dimensional is equivalent to (real) Hilbert space, while every finite-dimensional subspace is equivalent to Euclidean space of that dimension. Any complete normed linear space $T$ which has an inner product is characterized by its (finite or transfinite) cardinal "dimension-number" $n$. It is equivalent to the space of all sets $x = (x_1, x_2, \cdots)$ of $n$ real numbers satisfying $\sum x_i^2 < +\infty$, where $\|x\| = (\sum x_i^2)^{1/2}$ [7, Theorem 32]. Various necessary and sufficient conditions for the existence of an inner product in normed linear spaces of two or more dimensions are known. Two such conditions are that $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for all $x$ and $y$, and that $\lim_{n \to \infty} \|nx+y\| - \|nx\| = 0$ whenever $\|x\| = \|y\|$ ([5] and [4, Theorem 6.3]). A characterization of inner product spaces of three or more dimensions is that there exist a projection of unit norm on each two-dimensional subspace [6, Theorem 3]. Other characterizations valid for three or more dimensions will be given here, expressed by means of orthogonality, hyperplanes, and linear functionals.

A hyperplane of a normed linear space is any closed maximal linear subset $M$, or any translation $x+M$ of $M$. A hyperplane is a supporting hyperplane of a convex body $S$ if its distance from $S$ is zero and it does not contain an interior point of $S$; it is tangent to $S$ at $x$ if it is the only supporting hyperplane of $S$ containing $x$ [8, pp. 70-74]. It will be said that an element $x_0$ of $T$ is orthogonal to $y$ ($x_0 \perp y$) if and only if $\|x_0 + ky\| \geq \|x_0\|$ for all $k$, which is equivalent to requiring the existence of a nonzero linear functional $f$ such that $f(x_0) = \|f\| \|x_0\|$ and $f(y) = 0$, or that $x_0 + y$ belong to a supporting hyperplane of the sphere.

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1 Numbers in brackets refer to the references at the end of the paper.
2 "Equivalent" meaning isometric under a linear transformation [1, p. 180]. The equivalence to (real) Hilbert space follows by reasoning similar to that of [10, pp. 3-16].
\[ \|x\| \leq \|x_0\| \] at the point \( x_0 \) [4, Theorem 2.1 and §5]. In a space with an inner product, \( x \perp y \) if and only if \( (x, y) = 0 \).

Orthogonality is said to be additive on the right if and only if \( z \perp x \) and \( z \perp y \) imply \( z \perp x + y \). Clearly \( x \perp x \) implies \( x = 0 \), while \( x \perp y \) implies \( ax \perp by \) for any numbers \( a \) and \( b \). Every element is orthogonal to at least one hyperplane through the origin, this hyperplane being unique for any given element if and only if: (1) For any \( x \) \((\neq 0)\) and \( y \) there is a unique number \( a \) with \( x \perp ax + y \); (2) The unit sphere \( \|x\| \leq 1 \) of \( T \) has a tangent hyperplane at each point; (3) The norm is Gateaux differentiable; or (4) Orthogonality is additive on the right [4, Theorems 4.2, 5.1].

Orthogonality is said to be additive on the left if and only if \( x \perp z \) and \( y \perp z \) imply \( x + y \perp z \). Orthogonality is not symmetric in general, and there does not necessarily exist a hyperplane orthogonal to a given element (Theorems 1 and 5). Additivity on the left does not imply strict convexity;\(^3\) nor conversely, but a normed linear space is strictly convex if and only if: (1) For any \( x \) \((\neq 0)\) and \( y \) there is a unique number \( a \) with \( x + y \perp x \); or (2) No supporting hyperplane has more than one point of contact [4, Theorems 4.3, 5.2].

Birkhoff has shown that an inner product can be defined in a normed linear space of three or more dimensions if orthogonality is symmetric and unique.\(^4\) An equivalent condition is that \( N_+(x; y) = 0 \) whenever \( N_+(y; x) = 0 \), where \( N_+(x; y) = \lim_{h \to 0^+} \frac{\|x + hy\| - \|x\|}{h} \) exists because of the convexity of the function \( f(h) = \|x + hy\| \) [4, Theorem 6.2]. It is possible to show by a purely geometric argument that in a space of three or more dimensions orthogonality must be unique if it is symmetric, but this follows more easily from known facts about projections in normed linear spaces:

**Theorem 1.** Orthogonality is symmetric in a normed linear space \( T \) of three or more dimensions if and only if an inner product can be defined in \( T \).

**Proof.** Let \( x_1 \) and \( x_2 \) be any two elements of a three-dimensional subspace \( T_0 \) of \( T \). Then there is an element \( y \in T_0 \) orthogonal to the linear hull \( H_0 \) of \( x_1 \) and \( x_2 \) [4, Theorem 7.1]. If orthogonality is symmetric, then \( H_0 \perp y \). Hence if a projection of \( T_0 \) on \( H_0 \) is defined by \( z = P(z) + ay \), where \( P(z) \in H_0 \), then \( \|P(z)\| \leq \|z\| \) for all \( z \) and \( \|P\| = 1 \). But it is known that an inner product can be defined in a normed

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\(^3\) A normed linear space is strictly convex if \( \|x + y\| = \|x\| + \|y\| \) and \( y \neq 0 \) imply \( x = ty \) for some \( t \).

\(^4\) See [2]. With symmetry, uniqueness means the uniqueness for any \( x \) \((\neq 0)\) and \( y \) of the number \( a \) for which \( x \perp ax + y \).
linear space of three or more dimensions if there is a projection of norm one on any given closed linear subspace \[6, \text{Theorem 3}\]. Thus an inner product can be defined in any three-dimensional subspace of \(T\) and hence in \(T\) itself \[5\].

For elements \(x\) and \(y\) of a normed linear space, \(x \perp y\) if and only if there is a nonzero linear functional \(f\) such that \(f(x) = \|f\| \|x\|\) and \(f(y) = 0\), while \(ax + y \perp x\) if and only if \(\|kx + y\|\) is minimum for \(k = a\) \[4, \text{Theorems 2.1, 2.3}\]. Also, the set \(H\) of all \(z\) satisfying \(f(z) = \|f\|\|z\|\) is a supporting hyperplane of the unit sphere at \(x\) if \(f(x) = \|f\|\) and \(\|x\| = 1\), while any supporting hyperplane can be defined by such an equation (see Mazur \[8, \text{p. 71}\]). Also, \(H\) is said to be parallel to an element \(y\) if and only if \(f(y) = 0\) (that is, the line \(\{ky\}\) does not intersect \(H\)).

Interpretations of Theorem 1 by means of linear functionals and hyperplanes therefore give the following necessary and sufficient conditions for the existence of an inner product in a normed linear space of three or more dimensions:

1. For any elements \(x\) and \(y\), the existence of a nonzero linear functional \(f\) with \(f(x) = \|f\| \|x\|\) and \(f(y) = 0\) implies the existence of a nonzero linear functional \(g\) with \(g(y) = \|g\| \|y\|\) and \(g(x) = 0\).

2. For any elements \(x\) and \(y\), \(\|kx + y\|\) is minimum when \(k = -f(y)/f(x)\) if \(f\) is a linear functional with \(f(x) = \|f\| \|x\|\).

3. The existence of a supporting hyperplane of the unit sphere at \(x\) parallel to \(y\) (\(\|x\| = \|y\| = 1\)) implies the existence of a supporting hyperplane at \(y\) parallel to \(x\).

There are infinitely many different normed linear spaces of two dimensions in which orthogonality is not symmetric \[2, \text{Theorem 4}\]. If an isomorphism \(ax + by \leftrightarrow (a, b)\) is set up between the Cartesian plane and a two-dimensional normed linear space containing \(x\) and \(y\) (\(\|x\| = \|y\| = 1\)) and if \(C\) is the “unit pseudo-circle” of all points \((a, b)\) for which \(\|ax + by\| = 1\), then orthogonality is symmetric in \(T\) if and only if the line through the origin parallel to any supporting line of \(C\) at any point \(p\) cuts \(C\) in a point at which there is a supporting line parallel to the line from \(p\) to the origin. Let \(B_r\) \((r \geq 1)\) be the normed linear space of pairs \((x_1, x_2) = x\) of real numbers, where \(\|x\| = \left( |x_1| \cdot |x_2| \right)\) if \(x_1\) and \(x_2\) are of the same sign, and \(\|x\| = \left( |x_1| + |x_2| \right)\) otherwise, where \(s = r/(r - 1)\). It can easily be verified that orthogonality is symmetric in \(B_r\) for \(r \geq 1\), and that it is unique except in the limiting case \(r = 1\). Thus orthogonality can be symmetric and not unique in a two-dimensional space.

**Theorem 2.** An inner product can be defined in a normed linear space of three or more dimensions if and only if orthogonality is additive on the left.
PROOF. Let $T$ be a normed linear space of three or more dimensions, and $x_1$ and $x_2$ be any two elements. Then there are hyperplanes $H_1$ and $H_2$ with $x_1 \perp H_1$ and $x_2 \perp H_2$. Let $M = H_1 \cap H_2$. If orthogonality is additive on the left, then $ax_1 + bx_2 \perp M$ for all $a$ and $b$, and any element $z$ has a unique representation in the form $z = P(z) + y$, where $y \in M$ and $P(z) = ax_1 + bx_2$. Also, $\|z\| \geq \|P(z)\|$ for all $z$, and $\|P\| = 1$.

Since there is a projection of norm one on any given two-dimensional linear subspace of $T$, it follows as for Theorem 1 that an inner product can be defined in $T$ [6, Theorem 3].

The conclusion of the above theorem is not valid without the assumption that the space be of more than two dimensions, since it is clear that for a two-dimensional normed linear space orthogonality is additive on the left if and only if for any $x$ ($\neq 0$) there is a unique nonzero element orthogonal to $x$. It therefore follows that orthogonality is additive on the left in a two-dimensional normed linear space if and only if the space is strictly convex [4, Theorem 4.3].

If $L$ is a closed linear set in a Banach space $B$, then the normal projection of $x$ on $L$ is said to be the element $u$ for which $x - u \perp L$, or for which $\|x - u\|$ is the distance from $x$ to $L$. If $L$ is finite-dimensional, or if the unit sphere of $B$ is weakly compact, then normal projection is defined for all $x$ and $L$ [4, Theorem 7.2]. It was shown by Fortet [3, p. 45] that if orthogonality is symmetric in a uniformly convex Banach space, then normal projection is a continuous linear operation and the set $H$ of points $y$ with $y \perp x$ is linear and closed. However, it follows from the above theorems that $H$ is linear for all $x$ only if an inner product can be defined in the space $R$ and that the existence of an inner product follows from symmetry of orthogonality. Also, $x \perp L$ if and only if there is a linear functional $f$ with $f(x) = \|f\| \|x\|$ and $f(L) = 0$ [4, Theorem 2.1]. The following characterizations of inner product spaces of three or more dimensions are therefore direct consequences of Theorem 2.

(4) The existence of a linear functional $F$ with $F(x+y) = \|F\| \|x+y\|$ and $F(z) = 0$ whenever $x$, $y$, and $z$ are such that there are linear functionals $f$ and $g$ with $f(x) = \|f\| \|x\|$, $g(y) = \|g\| \|y\|$, and $f(z) = g(z) = 0$.

(5) That normal projection be a linear operation.

If a complete normed linear space has an inner product, then any linear functionals $f$ and $g$ can be written in the form $f(u) = (x, u)$ and $g(u) = (y, u)$, for some elements $x$ and $y$ [7, Theorem 11]. Then $F$ of (4) can be taken as $f+g$. For any linear functional $G = Af + Bg$, there are then numbers $a$ and $b$ such that $G(ax+by) = \|G\| \|ax+by\|$. This condition is also sufficient for an inner product:
Theorem 3. An inner product can be defined in a normed linear space \( T \) of three or more dimensions if and only if it follows from \( f(x) = \|f\| \|x\| \) and \( g(y) = \|g\| \|y\| \) for linear functionals \( f \) and \( g \) and elements \( x \) and \( y \) of \( T \) that there are numbers \( a \) and \( b \) such that \( f(ax+by) + g(ax+by) = \|f+g\| \|ax+by\| \) and \( ax+by \neq 0 \).

Proof. First note that if for some \( x \) there are two nonzero linear functionals \( F \) and \( G \) with \( F(x) = \|F\| \|x\| \) and \( G(x) = \|G\| \|x\| \), then the assumption of the theorem would imply that \( h(x) = \|h\| \|x\| \) if \( h = \|G\| F - \|F\| G \). But this is clearly impossible unless \( h \equiv 0 \), or \( \|G\| F = \|F\| G \). Thus two independent linear functionals cannot take on their maximum in the unit sphere \( \|x\| \leq 1 \) at the same point, which is known to imply that the unit sphere has a tangent hyperplane at each point [4, Theorem 5.1]. Now suppose that \( x \perp z \) and \( y \perp z \), and let \( T_0 \) be the linear hull of \( x, y, \) and \( z \). There are then two linear functionals \( f \) and \( g \) with \( f(x) = \|f\| \|x\| \), \( g(y) = \|g\| \|y\| \), and \( f(z) = g(z) = 0 \) [4, Theorem 2.1]. If \( x \) and \( y \) are not linearly independent, then \( x+\lambda z \). Let \( x \) and \( y \) be linearly independent and suppose that for \( u = x+y \) there are no numbers \( A \) and \( B \) satisfying \( Af(u) +Bg(u) \) \( = \|Af+Bg\| \|u\| \) and in \( C \) for which there are linear functionals \( f' = Af + Bg \) and \( g' = Af + Bg \) with \( f'(x') = \|f'\| \|x'\| \) and \( g'(y') = \|g'\| \|y'\| \), but such that none of the linear functionals \( Af' +Bg' \) satisfy \( Af'[x' + (1-r)y'] +Bg'[x' + (1-r)y'] = \|Af' +Bg'\| \|x' + (1-r)y'\| \) for any \( r \) with \( 0 < r < 1 \). For each such \( r \), there is a number \( \alpha_r \) for which \( [x' + (1-r)y'] + \alpha_r z] / \|x' + (1-r)y' + \alpha_r z\| = v \perp z \) [4, Theorem 2.3]. If \( h \) is a linear functional defined in \( T_0 \) for which \( h(v) = \|h\| \|v\| \) and \( h(z) = 0 \), and if \( A_r \) and \( B_r \) are such that \( Af'(s_0) +Bg'(s_0) = 0 \) for some \( s_0 \in T_0 \) for which \( h = 0 \) but not both \( f' \) and \( g' \) are zero, then \( h \) and \( Af' +Bg' \) are both zero at \( s_0 \) and \( z \) and hence are multiples of each other on \( T_0 \). Then if \( a_r \) and \( b_r \) are chosen by the assumptions of the theorem so that \( \|a_r x' + b_r y'\| = 1 \) and \( \|Af'(a_r x' + b_r y') +Bg'(a_r x' + b_r y')\| = \|Af' +Bg'\| \), it follows that \( h \) is a multiple of \( Af' +Bg' \) and that \( h(a_r x' + b_r y') = \|h\| \|a_r x' + b_r y'\| \). Thus the unit sphere \( S \) contains the straight lines \( l_r \) between \( a_r x' + b_r y' \) and \( v \), since the unit sphere is convex and the tangent hyperplane defined by \( h(x) = \|h\| \) contains \( a_r x' + b_r y' \) and \( v \). This tangent hyperplane at \( v \) then contains this line, but does not contain a point of \( C \) between \( x' \) and \( y' \). But there are also tangent hyperplanes at \( x' \) and \( y' \) parallel to \( z \), while \( a_r x' + b_r y' \) is by assumption not of the form \( [x' + (1-r)y'] / \|x' + (1-r)y'\| \) for any \( r \) satisfying \( 0 < r < 1 \). This implies that the tan-
tangent hyperplane at \( v \) contains either \( x' \) or \( y' \) and is coincident with the tangent hyperplane at \( x' \) or \( y' \), respectively. Letting \( r \) vary from 0 to 1, it now follows from the convexity of \( S \) that the tangent hyperplanes at \( x' \) and at \( y' \) have a common point of contact and must therefore coincide, since \( S \) has a tangent hyperplane at each point. This tangent hyperplane then contains the line from \( x' \) to \( y' \), and \( f'(x+y) = \| f' \| \| x+y \| \), contrary to assumption. Hence there are numbers \( A \) and \( B \) with \( Af(x+y) +Bg(x+y) = \| Af+Bg \| \| x+y \| \). Since \( Af(z) +Bg(z) = 0 \), this implies that \( x+y \perp z \) and that orthogonality is additive on the left. It now follows from Theorem 2 that an inner product can be defined in \( T \).

For any element \( x \) of a normed linear space there is always a hyperplane \( H \) through the origin with \( x \perp H \). However, for no hyperplane \( H \) of the space of continuous functions is there an element \( f \in C \) with \( H \perp f \). This follows from the fact that \( g \perp f \) if and only if \( \min_A gf \leq 0 \leq \max_A gf \), where \( A \) is the set of all \( t \) with \( |g(t)| = \| g \| \) [4, §4]. If \( T \) is one of the spaces \( (s), (m), (c) \), or \( l(p) (p \geq 1) \), then clearly \( H \perp x \) for an infinite number of different hyperplanes \( H \) and elements \( x \). If a normed linear space is strictly convex, then for no element \( x \) is there more than one hyperplane \( H \) with \( H \perp x \), while no hyperplane is orthogonal to more than one element if the norm of \( T \) is differentiable [4, Theorems 4.2, 4.3]. This difference is the reason for the lack of similarity between the proofs of the following theorems.

**Theorem 4.** An inner product can be defined in a normed linear space of three or more dimensions if and only if each hyperplane through the origin is orthogonal to at least one element.

**Proof.** Let \( x_1 \) and \( x_2 \) be any two elements of a normed linear space \( T \) of three or more dimensions, and let \( P_0 \) be the linear hull of \( x_1 \) and \( x_2 \). By well-ordering the set of all linear subspaces \( M \) of \( T \) for which \( P_0 \perp M \), it follows that there is a linear subspace \( \overline{M} \) of \( T \) such that \( P_0 \perp \overline{M} \) and \( \overline{M} \) is not contained properly in any other such linear subspace. Then it is clear that \( \overline{M} \) is closed. Hence if the linear hull \( H \) of \( P_0 \) and \( \overline{M} \) were not \( T \), there would be a hyperplane through the origin which contains \( P_0 \) and \( \overline{M} \). If every hyperplane through the origin is orthogonal to some element, then there would be an element \( x \) such that \( H \perp x \). But if \( y = x_p + x_m + kx \), where \( x_p \in P_0 \) and \( x_m \in \overline{M} \), then \( \| y \| \geq \| x_p + x_m \| \geq \| x_p \| \), since \( (x_p + x_m) \perp x \) and \( x_p \perp x_m \). Thus \( P_0 \) would be orthogonal to the linear hull of \( \overline{M} \) and \( x \). Hence the linear hull of \( P_0 \) and \( \overline{M} \) must be \( T \). A projection \( P(z) \) of \( T \) on \( P_0 \) can now be defined by \( z = P(z)+z_m \), where \( P(z) \in P_0 \) and \( z_m \in \overline{M} \). Since \( \| P \| = 1 \),
it follows that there is a projection of unit norm on any given two-dimensional linear subspace of $T$ and hence (as in the proof of Theorem 1) that an inner product can be defined in $T$.

**Theorem 5.** An inner product can be defined in a normed linear space $T$ of three or more dimensions if and only if for any $x \in T$ there is a hyperplane $H$ through the origin with $H \perp x$.

**Proof.** Suppose $x$, $y$, and $z$ are any three elements of $T$ with $x \perp z$ and $y \perp z$. If $T$ is strictly convex, then for any $u$ and $v$ of $T$ there is a unique $a$ such that $au + v \perp u$ [4, Theorem 4.3]. Hence if $H$ is a hyperplane through the origin with $H \perp z$, and if $T$ is strictly convex, then $x + y \in H$ and $x + y \perp z$, orthogonality is additive on the left, and an inner product can be defined in $T$. Now suppose $T$ is not strictly convex. Then there are elements $x$ and $y$ and a linear functional $f$ with $f(x) = f(y) = \|f\|$ and $\|x\| = \|y\| = 1$ [9, Theorem 6]. Let $z$ be any other element of unit norm not in the linear set generated by $x$ and $y$ and let $S_0$ be the unit sphere of the space $T_0$ generated by $x$, $y$, and $z$. Let $P_0$ be the set of all points $u \in S_0$ for which $\|u\| = 1$ and $f(u) = \|f\|$. Then $P_0$ contains the line from $x$ to $y$, and is itself either a straight line segment or a section of a plane.

Let $L_0$ be the hyperplane of $T_0$ with $P_0 \perp L_0$, where $L_0$ contains all points at which $f$ is zero. Then for any $v$ and each number $a$ there is a hyperplane $H_a$ of $T_0$ with $H_a \perp v + ax$. As $a \to 0$ (or as $a \to -0$), the planes $H_a$ will have at least one limit $H_+$ (or $H_-$) in the sense that there exist sequences $\{a_i\}$ and $\{b_i\}$, with $a_i \to +0$ and $b_i \to -0$, $\lim_{a_i \to 0} \rho(w, H_{a_i}) = 0$ and $\lim_{b_i \to 0} \rho(w, H_{b_i}) = 0$, if $w$ is any fixed element of $H_+$ or $H_-$, respectively. Since at each point of unit norm in $H_a$ there is a supporting plane of $S_0$ parallel to $v + ax$, it follows that if $v \in L_0$ then neither $H_+$ nor $H_-$ crosses $P_0$, and $P_0$ consists of those and only those points of the surface of $S_0$ in a region containing $x$ and bounded by $H_+$, $H_-$, and the two supporting lines of $P_0$ parallel to $v$. But this is possible for arbitrary $v \in L_0$ only if $P_0$ is a point.

Theorems 3–5 can be given direct interpretations by means of supporting hyperplanes of the unit sphere $S$, as was done for Theorem 1 to get (3). The first of these interpretations can be changed somewhat to give the following nontrivial consequence of Theorem 3.

**Theorem 6.** An inner product can be defined in a Banach space if every supporting hyperplane of the unit sphere $S$ has a point of contact and the existence of supporting hyperplanes $H_1$ and $H_2$ at points $x$ and $y$ of $S$ imply that any supporting hyperplane $H_3$ of $S$ satisfying $H_1 \cap H_2 \cap H_3 = 0$ have a point of contact which is in the linear hull of $x$ and $y$. 

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PROOF. First suppose that there is an element \( x \) and nonzero linear functionals \( f_1 \) and \( f_2 \) such that \( \|x\| = 1 \), \( f_1(x) = \|f_1\| \), and \( f_2(x) = \|f_2\| \). Then \( x \) is in both of the supporting hyperplanes \( H_1 \) and \( H_2 \) of \( S \), where \( H_1 \) and \( H_2 \) are defined by \( f_1(z) = \|f_1\| \) and \( f_2(z) = \|f_2\| \). If \( L \) is the set of points at which \( f_1 = f_2 = 0 \), then \( H_1 \cap H_2 = x + L \). If \( f_1 \) and \( f_2 \) are linearly independent, then the linear hull of \( x \) and \( L \) is not the whole space and there is a nonzero linear functional \( f_3 \) which is zero on \( x \) and \( L \). Let \( H_3 \) be defined by \( z \in H_3 \) if and only if \( f_3(z) = \|f_3\| \). Then clearly \( H_1 \cap H_2 \cap H_3 = 0 \). But the second hypothesis of the theorem would imply that \( x \in H_3 \), or \( f_3(x) = \|f_3\| \), which contradicts \( f_3(x) = 0 \). Therefore \( f_1 \) and \( f_2 \) are linearly dependent.

Now suppose that \( \|x\| = \|y\| = 1 \), \( f_1(x) = \|f_1\| \), and \( f_2(y) = \|f_2\| \). If \( f_3 = f_1 + f_2 \), and \( H_1 \), \( H_2 \), \( H_3 \) are defined by \( f_i(z) = \|f_i\| \) \((i = 1, 2, 3)\), then \( x \in H_1 \) and \( y \in H_2 \). If \( H_1 \cap H_2 \cap H_3 \neq 0 \), then there exists an element \( w \) such that \( f_1(w) = \|f_1\| \), \( f_2(w) = \|f_2\| \), and \( f_1(w) + f_2(w) = \|f_1 + f_2\| \). Thus \( \|f_1 + f_2\| = \|f_1\| + \|f_2\| \). Since every linear functional in \( T \) takes on its maximum in the unit sphere, \( H_3 \) contains a point \( z \) of norm 1. Then \( f_1(z) + f_2(z) = \|f_1 + f_2\| = \|f_1\| + \|f_2\| \). Therefore \( f_1(z) = \|f_1\| \) and \( f_2(z) = \|f_2\| \). Hence \( f_1 \) and \( f_2 \) must be linearly dependent, and \( f_1(z) + f_2(z) = \|f_1 + f_2\| \). If \( H_1 \cap H_2 \cap H_3 = 0 \), then \( H_2 \) has a point of contact \( ax + by \) \((\|ax + by\| = 1) \) and \( f_3(ax + by) = \|f_3\| \|ax + by\| \). Thus it follows from Theorem 3 that an inner product can be defined.

REFERENCES


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