CONCERNING AUTOMORPHISMS OF NON-ASSOCIATIVE ALGEBRAS

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In their studies of non-associative algebras A. A. Albert and N. Jacobson have made much use of the relationships which exist between an arbitrary non-associative algebra \( A \) and its associative transformation algebra \( T(\mathbb{A}) \). In this paper we are interested in the automorphism group \( \mathcal{G} \) of \( A \), and we sharpen the results of Jacobson [3, §4] and Albert [2, §9] in the sense that we prove \( \mathcal{G} \) isomorphic to a well-defined subgroup of the automorphism group of each of three associative algebras (§§2, 3).

Incidental to our proofs is the reconstruction (in the sense of equivalence) of an arbitrary non-associative algebra \( A \) with unity element 1 from \( T(\mathbb{A}) \) and from either of the enveloping algebras \( E(R(\mathbb{A})) \), \( E(L(\mathbb{A})) \) of respectively the right or left multiplications of \( A \). This paper has been expanded in accordance with suggestions of the referee to include a more detailed study of the right ideals used in this reconstruction process (§5).

1. Preliminaries. Our notations are chiefly those of Albert as given in [1]. We regard a non-associative algebra \( A \) of order \( n \) over a field \( \mathbb{F} \) as consisting of a linear space \( \mathcal{L} \) of order \( n \) over \( \mathbb{F} \), a linear space \( R(\mathbb{A}) \) of linear transformations \( R_\cdot \) on \( \mathcal{L} \) of order \( m \leq n \) over \( \mathbb{F} \), and a linear mapping of \( \mathcal{L} \) on \( R(\mathbb{A}) \),

\[
(1) \quad x \rightarrow R_\cdot.
\]

The elements \( R_\cdot \) of \( R(\mathbb{A}) \) are called right multiplications, and \( R(\mathbb{A}) \) the right multiplication space of \( A \). Multiplication in \( A \) is defined by

\[
(2) \quad a \cdot x = aR_\cdot.
\]

The linearity of the right multiplications and of (1) insures distributivity in \( A \) as well as the usual laws of scalar multiplication. We shall use the fact that, in case \( \mathbb{A} \) contains no absolute right divisor of zero (an element \( x \) such that \( a \cdot x = 0 \) for all \( a \) in \( A \)), the mapping (1) is nonsingular and the order of \( R(\mathbb{A}) \) over \( \mathbb{F} \) is \( n \).

The linear transformations \( L_a \) defined by

\[
(3) \quad a \rightarrow x \cdot a = aL_\cdot
\]
are called left multiplications of \( \mathfrak{A} \) and form the left multiplication space \( L(\mathfrak{A}) \) of \( \mathfrak{A} \). The algebra \( \mathfrak{A} \) may equally well be regarded as consisting of \( \mathfrak{g} \), \( L(\mathfrak{A}) \), and the linear mapping

\[
(4) \quad x \rightarrow L_x
\]

of \( \mathfrak{g} \) on \( L(\mathfrak{A}) \). Both \( R(\mathfrak{A}) \) and \( L(\mathfrak{A}) \) are linear subspaces of the total matric algebra \( (\mathfrak{g})_n \) of all linear transformations on \( \mathfrak{g} \).

If \( \mathfrak{M} \) is a subset of \( (\mathfrak{g})_n \), the algebra of all polynomials in the transformations in \( \mathfrak{M} \) with coefficients in \( \mathfrak{g} \) is called the enveloping algebra of \( \mathfrak{M} \), and is denoted by \( E(\mathfrak{M}) \). We are particularly concerned with the enveloping algebras \( E(R(\mathfrak{A})) \) and \( E(L(\mathfrak{A})) \) of respectively the right and left multiplications of \( \mathfrak{A} \), and with the transformation algebra \( T(\mathfrak{A}) = E(I, R(\mathfrak{A}), L(\mathfrak{A})) \) which is the algebra of all polynomials with coefficients in \( \mathfrak{g} \) in the right and left multiplications of \( \mathfrak{A} \) and the identity transformation \( I \) in \( (\mathfrak{g})_n \). We shall have occasion to write an arbitrary element \( T \) of each of these algebras as follows:

\[
(5) \quad T = f(R_x, R_y, \ldots) \quad \text{for } T \text{ in } E(R(\mathfrak{A})),
\]

\[
(6) \quad T = f(L_z, L_y, \ldots) \quad \text{for } T \text{ in } E(L(\mathfrak{A})),
\]

\[
(7) \quad T = f(I, R_x, L_z, R_y, \ldots) \quad \text{for } T \text{ in } T(\mathfrak{A}),
\]

where \( x, y, \ldots \) are elements of \( \mathfrak{A} \). In case \( \mathfrak{A} \) contains a unity element \( 1 \), then \( R(\mathfrak{A}) \) contains \( I \), and we may write

\[
(8) \quad T = f(R_x, L_z, R_y, \ldots) \quad \text{for } T \text{ in } T(\mathfrak{A}),
\]

\( x, y, \ldots \) in \( \mathfrak{A} \).

If \( \mathfrak{B} \) is a linear subspace of \( \mathfrak{A} \), the set of all \( R_b \) for \( b \) in \( \mathfrak{B} \) is a linear subspace \( R(\mathfrak{B}, \mathfrak{A}) \) of \( R(\mathfrak{A}) \), and the set of all \( L_b \) is a linear subspace \( L(\mathfrak{B}, \mathfrak{A}) \) of \( L(\mathfrak{A}) \).

An automorphism \( S \) of an algebra \( \mathfrak{A} \) is a nonsingular linear transformation \( x \rightarrow xS \) of \( \mathfrak{A} \) on itself such that

\[
(9) \quad (a \cdot x)S = aS \cdot xS
\]

for all \( a, x \) in \( \mathfrak{A} \). In terms of right and left multiplications, (9) may be written equivalently as

\[
(10) \quad R_xS = SR_xS
\]

or

\[
(11) \quad L_xS = SL_xS
\]

for all \( x \) in \( \mathfrak{A} \). We shall use the facts that, if \( S \) is an automorphism of \( \mathfrak{A} \), then \( S^{-1} \) is also, and if \( \mathfrak{A} \) has a unity element \( 1 \), then \( 1S = 1 \).
Inasmuch as the elements $T$ of subalgebras of $(\mathfrak{A})_n$ are themselves linear transformations, we shall denote linear transformations on subalgebras of $(\mathfrak{A})_n$—such as $T(\mathfrak{A})$, $E(R(\mathfrak{A}))$, $E(L(\mathfrak{A}))$—by Greek capitals, so that if $\Sigma$ is a linear transformation on $T(\mathfrak{A})$, say, we may write (without confusion) the image of $T$ under $\Sigma$ as $T\Sigma$.

An automorphism $S$ of $\mathfrak{A}$ determines an automorphism $\Sigma$ of $T(\mathfrak{A})$ as follows: let $T$ in $T(\mathfrak{A})$ be written in the form (7); then $\Sigma$ is the mapping

$$ T \rightarrow T \Sigma = f(I, R_x, L_x, R_y, \cdots) = S^{-1}TS. $$

For if $S$ is an automorphism of $\mathfrak{A}$, then $R_x = SR_xS^{-1}$, $L_x = SL_xS^{-1}$ by (10), (11), and $T = f(I, R_x, L_x, R_y, \cdots) = f(I, SR_xS^{-1}, SL_xS^{-1}, SR_yS^{-1}, \ldots) = Sf(I, R_x, L_x, R_y, \cdots)S^{-1} = S(T\Sigma)S^{-1}$, or $T\Sigma = S^{-1}TS$. The mapping (12) is obviously an automorphism of $T(\mathfrak{A})$.

Moreover, $\Sigma$ induces automorphisms (which we do not distinguish notationally from $\Sigma$) on the subalgebras $E(R(\mathfrak{A}))$, $E(L(\mathfrak{A}))$ of $T(\mathfrak{A})$:

(13) $T \rightarrow T \Sigma = f(R_x, R_y, \cdots)$, $T$ in $E(R(\mathfrak{A}))$ as in (5),

(14) $T \rightarrow T \Sigma = f(L_x, L_y, \cdots)$, $T$ in $E(L(\mathfrak{A}))$ as in (6).

If $S$ determines $\Sigma$ as in (12), then

$$ R(\mathfrak{A}) \Sigma = R(\mathfrak{A}), \quad L(\mathfrak{A}) \Sigma = L(\mathfrak{A}), $$

since $R_x \Sigma = R_x$ in $R(\mathfrak{A})$ while $L_x \Sigma = L_x$ in $L(\mathfrak{A})$, and the nonsingularity of $\Sigma$ eliminates the possibility of proper inclusion.

2. Automorphisms of an algebra with unity element. Let $\mathfrak{A}$ be a non-associative algebra of order $n$ over $\mathfrak{F}$ with unity element 1. We consider the elements of $\mathfrak{A}$ as comprising a linear space $\mathfrak{F}$ of order $n$ over $\mathfrak{F}$. Let $\mathfrak{B}$ be any (associative) algebra of linear transformations on $\mathfrak{F}$ which contains either $R(\mathfrak{A})$ or $L(\mathfrak{A})$. We intend to reconstruct $\mathfrak{A}$ (in the sense of equivalence) as an algebra of residue classes of $\mathfrak{B}$.

Denote by $\mathfrak{N}$ the set of all transformations $N$ in $\mathfrak{B}$ which annihilate 1, that is, for which $1N = 0$. Then $\mathfrak{N}$ is a right ideal of $\mathfrak{B}$. For if $N$, $N_1$ are in $\mathfrak{N}$, then $1(\alpha N + \beta N_1) = \alpha 1N + \beta 1N_1 = 0$ for $\alpha$, $\beta$ in $\mathfrak{F}$, while $1NT = 0T = 0$ for any transformation $T$ in $\mathfrak{B}$. Denote by $\mathfrak{D}$ whichever set $R(\mathfrak{A})$ or $L(\mathfrak{A})$ is assumed to be contained in $\mathfrak{B}$, and by $D_x$ correspondingly the transformation $R_x$ or $L_x$. Then $\mathfrak{B}$ is the supplementary sum $\mathfrak{B} = \mathfrak{D} + \mathfrak{N}$. For $T$ in $\mathfrak{B}$ may be written uniquely in the form $T = D_t + N$, $1T = t$, $N$ in $\mathfrak{N}$.

Since $1TN = tN$ which is not necessarily zero, $\mathfrak{N}$ is not in general a two-sided ideal of $\mathfrak{B}$ and we are not able to form the difference algebra $\mathfrak{B} - \mathfrak{N}$ when we take residue classes $[T]$ modulo $\mathfrak{N}$. Instead we form
the difference group $\mathfrak{B} - \mathfrak{N}$ of residue classes $[T]$ modulo $\mathfrak{N}$ and have as usual a linear set over $\mathfrak{F}$ with respect to the operations $[T] + [U] = [T + U]$, $\lambda [T] = [\lambda T]$ of addition and scalar multiplication. Define multiplication in this linear set as follows:

\[(16)\quad [T][U] = [X] \quad \text{where} \quad 1X = 1T \cdot U,\]

where the multiplication on the right is that in $\mathfrak{A}$. To see that for any $T, U$ in $\mathfrak{B}$ such an $X$ exists, we need only to note that, if $X = D_x$ for $x = 1T \cdot 1U$ in $\mathfrak{A}$, then $1X = 1T \cdot 1U$. This definition of multiplication is independent of the representatives $T, U$ since if $[T] = [T_1]$, $[U] = [U_1]$, then there exist $N, N_1$ in $\mathfrak{N}$ such that $T_1 = T + N$, $U_1 = U + N_1$, and $1T_1 \cdot 1U_1 = 1T \cdot 1U$. With this multiplication the distributive laws hold in $\mathfrak{B} - 91$. Hence $\mathfrak{B} - 91$ is a non-associative algebra over $\mathfrak{F}$. Since there are no difference algebras used in this paper, there should be no confusion in the use of the notation $\mathfrak{B} - 91$ for this algebra with multiplication defined by (16).

**Theorem 1.** Let $\mathfrak{A}$ be a non-associative algebra over $\mathfrak{F}$ with unity element 1, and $\mathfrak{B}$ be any (associative) algebra of linear transformations on $\mathfrak{A}$ containing either $\mathfrak{R}(\mathfrak{A})$ or $\mathfrak{L}(\mathfrak{A})$. If $\mathfrak{N}$ is the right ideal of transformations in $\mathfrak{B}$ annihilating 1, then the non-associative algebra $\mathfrak{B} - 91$ with multiplication defined by (16) is equivalent to $\mathfrak{A}$.

For each residue class $[T]$ there is a unique transformation $D_x$ in $\mathfrak{B}$ ($= \mathfrak{R}(\mathfrak{A})$ or $\mathfrak{L}(\mathfrak{A})$) such that $1T = 1D_x = t$. Then, since $\mathfrak{A}$ contains neither absolute right nor absolute left divisors of zero, the (obviously linear) mapping

\[(17)\quad x \rightarrow D_x \rightarrow [D_x]\]

is one-to-one on $\mathfrak{A}$ to $\mathfrak{B} - 91$. But

\[(18)\quad [D_x][D_y] = [D_{xy}], \quad x, y \in \mathfrak{A},\]

since $xy = 1D_x \cdot 1D_y = 1D_{xy}$. Then (17) is an equivalence of $\mathfrak{A}$ and $\mathfrak{B} - 91$ since $xy = D_{xy} \rightarrow [D_{xy}] = [D_x][D_y]$ under (17).

Now $\mathfrak{T}(\mathfrak{A})$, $\mathfrak{E}(\mathfrak{A})$, $\mathfrak{B}(\mathfrak{A})$ are among the algebras of linear transformations on the vector space $\mathfrak{F}$ underlying $\mathfrak{A}$ which contain either $\mathfrak{R}(\mathfrak{A})$ or $\mathfrak{L}(\mathfrak{A})$—or both, as in the case of $\mathfrak{T}(\mathfrak{A})$—and may be used as the algebra $\mathfrak{B}$ in Theorem 1. We denote by $\mathfrak{N}_T$ the set of all $N$ in $\mathfrak{T}(\mathfrak{A})$ annihilating 1 and write $\mathfrak{N}_T = \mathfrak{N}_T \cap \mathfrak{E}(\mathfrak{A})$, $\mathfrak{N}_L = \mathfrak{N}_T \cap \mathfrak{E}(\mathfrak{A})$. Then Theorem 1 implies that if multiplication in the respective algebras of residue classes is defined by (16) we have $\mathfrak{A} \cong \mathfrak{T}(\mathfrak{A}) - \mathfrak{N}_T \cong \mathfrak{E}(\mathfrak{A}) - \mathfrak{N}_T \cong \mathfrak{E}(\mathfrak{A}) - \mathfrak{N}_L$.

In the proof of the next theorem we must distinguish between the
cases $\mathfrak{D} = R(\mathfrak{A})$ and $\mathfrak{D} = L(\mathfrak{A})$, and we use the following equations:

(19) \[ [R_x][R_y] = [R_{xy}], \quad x, y \in \mathfrak{A}, \]

(20) \[ [L_x][L_y] = [L_{xy}], \quad x, y \in \mathfrak{A}, \]

verification of which is similar to that of (18).

**Theorem 2.** Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{S}$ be as in Theorem 1, and $\mathfrak{D}$ be $R(\mathfrak{A})$ or $L(\mathfrak{A})$, whichever is assumed to be in $\mathfrak{B}$. If $\Sigma$ is an automorphism of $\mathfrak{B}$ such that $\mathfrak{B} \Sigma = \mathfrak{S}$ and $\mathfrak{D} \Sigma = \mathfrak{D}$, then $\Sigma$ determines an automorphism $S_\mathfrak{D}$ of $\mathfrak{A}$ as follows:

(21) \[ S_\mathfrak{D}: \ x \to [D_x] \to [D_x \Sigma] = [D_x'] \to x' = x S_\mathfrak{D}, \]

for $x, x'$ in $\mathfrak{A}$, where the $[D_x]$ are elements of $\mathfrak{B} - \mathfrak{S} \cong \mathfrak{A}$.

Note first that the mapping

(22) \[ [T] \to [T \Sigma] \]

of $\mathfrak{B} - \mathfrak{S}$ on itself is well-defined, since if $[T] = [T_1]$ then $T = T_1 + N$ for $N$ in $\mathfrak{S}$, and $T \Sigma = (T_1 + N) \Sigma = T_1 \Sigma + N \Sigma = T_1 + N_1$ with $N_1$ in $\mathfrak{S}$ since $\mathfrak{S} \Sigma = \mathfrak{S}$. Hence $[T \Sigma] = [T_1 \Sigma]$. Inasmuch as the correspondences $x \to [D_x]$ and $[D_x'] \to x'$ are equivalences between $\mathfrak{A}$ and $\mathfrak{B} - \mathfrak{S}$, we need only to show that (22) is an automorphism of $\mathfrak{B} - \mathfrak{S}$ in order to show that (21) is an automorphism of $\mathfrak{A}$. Now (22) is linear since $\alpha[T] + \beta[U] = [\alpha T + \beta U] \to [(\alpha T + \beta U) \Sigma] = [\alpha T \Sigma + \beta U \Sigma] = \alpha[T \Sigma] + \beta[U \Sigma]$, and is nonsingular since $[T] \to [T \Sigma] = [0]$ implies $T \Sigma = N$ in $\mathfrak{S}$, $T = N \Sigma^{-1} = N_1$ in $\mathfrak{S}$, $[T] = [0]$. Since $\mathfrak{D} \Sigma = \mathfrak{D}$, there exists an element $x_1$ of $\mathfrak{A}$ such that $D_{x_1} \Sigma = D_{x_1}$. But then $x_1 = x'$ since there is a unique transformation in $\mathfrak{S}$ in each residue class of $\mathfrak{B}$ modulo $\mathfrak{S}$. We may write $x' = x S_\mathfrak{D}$ and

(23) \[ D_x \Sigma = D_x s_{x_1}. \]

We distinguish now between the cases $\mathfrak{D} = R(\mathfrak{A})$ and $\mathfrak{D} = L(\mathfrak{A})$. Let $\mathfrak{D} = R(\mathfrak{A})$ so that (19) holds. Then, since $\Sigma$ is an automorphism of $\mathfrak{B}$, we have $[R_x][R_y] = [R_{xy}] \to [(R_x)(R_y) \Sigma] = [(R_x \Sigma)(R_y \Sigma)] = [R_{xy} s_{x_1} s_{y_1}]$ under (22) which is an automorphism of $\mathfrak{B} - \mathfrak{S}$ as desired. In case $\mathfrak{D} = L(\mathfrak{A})$ it follows from (20) that $[L_x][L_y] = [L_y L_x] \to [(L_y L_x) \Sigma] = [(L_y \Sigma)(L_x \Sigma)] = [L_y s_{x_1} L_x s_{y_1}]$ under (22), completing the proof of the theorem.

We shall have occasion in the proof of the next theorem to use the fact that if $\mathfrak{B}$ contains both $R(\mathfrak{A})$ and $L(\mathfrak{A})$, and if both $R(\mathfrak{A})$ and $L(\mathfrak{A})$—as well of course as $\mathfrak{S}$—are their own images under an automorphism $\Sigma$ of $\mathfrak{B}$, then
(24) \[ R_x \Sigma = R_x s_x, \quad L_x \Sigma = L_x s_x \]
for \( S_x \) defined by (21).

**Theorem 3.** Let \( \mathfrak{A} \) be a non-associative algebra with unity element 1 and automorphism group \( \mathfrak{S} \). Let \( \mathfrak{S}_T \) be the group of automorphisms \( \Sigma \) of \( T(\mathfrak{A}) \) such that \( R_T \Sigma = R_T, \ R(\mathfrak{A}) \Sigma = R(\mathfrak{A}), \ L(\mathfrak{A}) \Sigma = L(\mathfrak{A}) \). Then the correspondence \( S \rightarrow \Sigma \) of (12) is an isomorphism of \( \mathfrak{S} \) onto \( \mathfrak{S}_T \).

If \( S \) is in \( \mathfrak{S} \) and \( S \rightarrow \Sigma \) under (12), then \( 1(NS) = 1S^{-1}NS = 1NS = 0S = 0 \) for \( N \) in \( R_T \). The nonsingularity of \( \Sigma \) gives \( R_T \Sigma = R_T \). By (15) we have \( \Sigma \) in \( \mathfrak{S}_T \). By Theorem 2 this \( \Sigma \) determines an automorphism \( S_2 \) of \( \mathfrak{A} \):

\[ S_2 : \quad x \rightarrow [R_x] \rightarrow [R_x \Sigma] = [R_x s_x] \rightarrow xS = xS_2 \]
for all \( x \) in \( \mathfrak{A} \), or \( S = S_2 \). Conversely, let \( \Sigma \) be in \( \mathfrak{S}_T \). Then \( \Sigma \) determines an automorphism \( S_2 \) of \( \mathfrak{A} \) which in turn determines an automorphism

(25) \[ \Sigma_2 : \quad T \rightarrow T \Sigma_2 = S_2^{-1}TS_2, \quad T \in T(\mathfrak{A}), \]
of \( T(\mathfrak{A}) \) by (12). Write \( T \) in the form (8). Then \( T \Sigma_2 = f(R_x s_x, L_x s_x, R_y s_y, \ldots) = f(R_x \Sigma, L_x \Sigma, R_y \Sigma, \ldots) = \{f(R_x, L_x, R_y, \ldots)\} \Sigma = T \Sigma \)
by (12), (24), and the fact that \( \Sigma \) is an automorphism of \( T(\mathfrak{A}) \). That is, \( \Sigma_2 = \Sigma \). It is clear then that \( S \rightarrow \Sigma \) is a one-to-one mapping of \( \mathfrak{S} \) onto \( \mathfrak{S}_T \). To see that \( S \rightarrow \Sigma \) is an isomorphism we note only that if \( S_1, S_2 \) are in \( \mathfrak{S} \), \( S_1 \rightarrow \Sigma_1, S_2 \rightarrow \Sigma_2 \), then for \( T \) in \( T(\mathfrak{A}) \) we have \( T \Sigma_1 = S_1^{-1}TS_1, T \Sigma_2 = S_2^{-1}S_2^{-1}TS_2 = (S_1S_2)^{-1}T(S_1S_2) \), or \( S_1S_2 \rightarrow \Sigma_1 \Sigma_2 \).

Variations in the proof of the following theorem from the proof above are trivial, consisting only of changes due to the fact that elements of \( E(R(\mathfrak{A})) \) or \( E(L(\mathfrak{A})) \) are generated by right or left multiplications alone.

**Theorem 4.** The correspondences \( S \rightarrow \Sigma \) of (13) and (14) are isomorphisms of \( \mathfrak{S} \) onto \( \mathfrak{S}_R \) and \( \mathfrak{S}_L \) respectively, where \( \mathfrak{S}_R \) is the group of automorphisms \( \Sigma \) of \( E(R(\mathfrak{A})) \) such that \( R_R \Sigma = R_R, R(\mathfrak{A}) \Sigma = R(\mathfrak{A}) \), and \( \mathfrak{S}_L \) is the group of automorphisms \( \Sigma \) of \( E(L(\mathfrak{A})) \) such that \( R_L \Sigma = R_L, L(\mathfrak{A}) \Sigma = L(\mathfrak{A}) \).

3. **Automorphisms of an algebra without unity element.** In case we are concerned with an algebra \( \mathfrak{A}_0 \) of order \( n - 1 \) over \( \mathfrak{F} \) without a unity element, we can easily modify the results of §2 to include \( \mathfrak{A}_0 \).

For we adjoin a unity element 1 to \( \mathfrak{A}_0 \) in the usual fashion to obtain an algebra \( \mathfrak{A} \) of order \( n \) over \( \mathfrak{F} \) containing \( \mathfrak{A}_0 \) (in the sense of equivalence) as an ideal. Every element \( x \) of \( \mathfrak{A} \) may be written uniquely in the form
\[(26) \quad x = \xi 1 + x_0, \quad \xi \text{ in } \mathfrak{g}, \quad x_0 \text{ in } \mathfrak{u}_0,\]

and if \(y = \eta 1 + y_0\), then \(x + y = (\xi + \eta) 1 + (x_0 + y_0)\), \(\delta x = (\delta \xi) 1 + (\delta x_0)\) for \(\delta\) in \(\mathfrak{g}\), \(xy = (\xi \eta) 1 + (\eta x_0 + \xi y_0 + x_0 y_0)\). We shall write \(\mathfrak{u} = \mathfrak{g} + \mathfrak{u}_0\) for the algebra so defined. Any automorphism \(S_0\) of \(\mathfrak{u}_0\) may be extended in a unique fashion to an automorphism \(S\) of \(\mathfrak{u}\) by defining
\[(27) \quad S: \quad x \to xS = \xi 1 + x_0 S_0,\]

\(x\) as in \((26)\). Note that \(S\) induces the automorphism \(S_0\) within \(\mathfrak{u}_0\).

It is apparent that an automorphism \(S_0\) of \(\mathfrak{u}_0\) determines a unique automorphism \(\Sigma\) of \(T(\mathfrak{u})\) as follows: \(S_0 \to S\) by \((27)\), \(S \to \Sigma\) by \((12)\). Moreover, the linear subspaces \(R(\mathfrak{u}_0, \mathfrak{u})\) and \(L(\mathfrak{u}_0, \mathfrak{u})\) of \(T(\mathfrak{u})\) are their own images under \(\Sigma\). For if \(x_0\) is in \(\mathfrak{u}_0\), then \(R_{x_0} \Sigma = R_{x_0} S = R_{x_0} S_0\) is in \(R(\mathfrak{u}_0, \mathfrak{u})\) and \(L_{x_0} \Sigma = L_{x_0} S = L_{x_0} S_0\) is in \(L(\mathfrak{u}_0, \mathfrak{u})\).

If \(\mathfrak{B} - \mathfrak{R}\) is the non-associative algebra equivalent to \(\mathfrak{u}\) which was defined in \(\S 2\), then \(\mathfrak{u}_0\) is equivalent to the ideal \(\mathfrak{g}_0\) of \(\mathfrak{B} - \mathfrak{R}\) consisting of residue classes \([D_{x_0}]\) for \(x_0\) in \(\mathfrak{u}_0\), that is, for \(D_{x_0}\) in \(\mathfrak{D}_0 = R(\mathfrak{u}_0, \mathfrak{u})\) or \(L(\mathfrak{u}_0, \mathfrak{u})\) according as \(\Sigma = R(\mathfrak{u})\) or \(L(\mathfrak{u})\). For, by Theorem 1, \(\mathfrak{u}\) is isomorphic to \(\mathfrak{B} - \mathfrak{R}\) under the mapping \((17)\). Since \(\mathfrak{u}_0\) is an ideal of \(\mathfrak{u}\), the mapping
\[(28) \quad x_0 \to [D_{x_0}], \quad x_0 \text{ in } \mathfrak{u}_0,\]
determines an ideal \(\mathfrak{g}_0\) of \(\mathfrak{B} - \mathfrak{R}\), and \(\mathfrak{g}_0 \leq \mathfrak{u}_0\).

Let \(\Sigma\) be an automorphism of \(\mathfrak{B}\) such that \(\mathfrak{B} \Sigma = \mathfrak{B}\) and \(\mathfrak{D}_0 \Sigma = \mathfrak{D}_0\). Then \(\Sigma\) determines an automorphism \(S_{\Sigma}\) of \(\mathfrak{g}_0\) as follows:
\[(29) \quad S_{\Sigma}: \quad x_0 \to [D_{x_0}] \to [D_{x_0} \Sigma] = [D_{x_0'}] \to x_0' = x_0 S_{\Sigma},\]
for \(x_0, x_0'\) in \(\mathfrak{u}_0\). For \(\mathfrak{D} = I\mathfrak{g} + \mathfrak{D}_0\), and any automorphism of \(\mathfrak{B}\) leaves invariant the subspace \(I\mathfrak{g}\) of order 1, so that \(\mathfrak{D} \Sigma = \mathfrak{D}\). Then by Theorem 2, \(\Sigma\) determines an automorphism \(S_{\Sigma}\) of \(\mathfrak{B}\). But \(S_{\Sigma}\) induces on \(\mathfrak{u}_0\) the automorphism \((29)\) since \(D_{x_0} \Sigma = D_{x_0'}\) in \(\mathfrak{D}_0\) implies \(x_0'\) is in \(\mathfrak{u}_0\). Thus \(x_0 \to [D_{x_0}] \to [D_{x_0} \Sigma] = [D_{x_0'}] \to x_0' = x_0 S_{\Sigma}\) is in \(\mathfrak{u}_0\), or \(S_{\Sigma}\) induces \(S_{\Sigma}\) on \(\mathfrak{u}_0\).

**Theorem 5.** Let \(\mathfrak{u}_0\) be a non-associative algebra without unity element, and let \(\mathfrak{u} = I\mathfrak{g} + \mathfrak{u}_0\). Let \(S_\Sigma\) be the group of automorphisms \(\Sigma\) of \(T(\mathfrak{u})\) such that \(R(\mathfrak{u}_0, \mathfrak{u}) \Sigma = R(\mathfrak{u}_0, \mathfrak{u}), L(\mathfrak{u}_0, \mathfrak{u}) \Sigma = L(\mathfrak{u}_0, \mathfrak{u}), \mathfrak{u}_T \Sigma = \mathfrak{u}_T\). Then the correspondence \(S_0 \to S \to \Sigma\) of \((27)\) and \((12)\) is an isomorphism of the automorphism group \(S_0\) of \(\mathfrak{u}_0\) onto \(S_\Sigma\).

For if \(S_0\) is in \(S_0\), then \(S_0 \to S \to \Sigma\) in \(S_\Sigma\) and \(\Sigma \to S_{\Sigma} = S\) by Theorem 3. But then \(S\) induces the automorphism \(S_{\Sigma}\) within \(\mathfrak{u}_0\). That is, \(S_{\Sigma} = S_0\). Conversely, if \(\Sigma\) is in \(S_\Sigma\), then \(\Sigma \to S_{\Sigma}\) in \(S_0\) by \((29)\). But \(S_{\Sigma} \to S_{\Sigma} \to \Sigma_{\Sigma}\) by \((27)\) and \((12)\) and \(\Sigma_{\Sigma} = \Sigma\) by Theorem 3. Hence the
mapping \( S_0 \mapsto S \mapsto \Sigma \) of \( \mathfrak{g}_0 \) on \( \mathfrak{g}'_0 \) is one-to-one, and is by Theorem 3 an isomorphism.

The results analogous to Theorem 4 for algebras \( \mathfrak{A}_0 \) without unity quantity may be stated as follows: let \( \mathfrak{g}'_0 \) be the group of automorphisms \( \Sigma \) of \( E(R(\mathfrak{A})) \) such that \( R(\mathfrak{A}_0, \mathfrak{A}) \Sigma = R(\mathfrak{A}_0, \mathfrak{A}) \), \( \mathfrak{g}'_R \Sigma = \mathfrak{g}'_R \). Then the correspondence \( S_0 \mapsto S \mapsto \Sigma \) of (27) and (13) is an isomorphism of the automorphism group \( \mathfrak{g}_0 \) of \( \mathfrak{A}_0 \) onto \( \mathfrak{g}'_0 \). Let \( \mathfrak{g}'_L \) be the group of automorphisms \( \Sigma \) of \( E(L(\mathfrak{A})) \) such that \( L(\mathfrak{A}_0, \mathfrak{A}) \Sigma = L(\mathfrak{A}_0, \mathfrak{A}) \), \( \mathfrak{g}'_L \Sigma = \mathfrak{g}'_L \). Then the correspondence \( S_0 \mapsto S \mapsto \Sigma \) of (27) and (14) is an isomorphism of \( \mathfrak{g}_0 \) onto \( \mathfrak{g}'_L \).

4. Inner automorphisms \( \Sigma \) of \( T(\mathfrak{A}) \). An automorphism \( \Sigma \) of the associative algebra \( T(\mathfrak{A}) \) is called inner in case \( T \mapsto T \Sigma = K^{-1}TK \) for some nonsingular element \( K \) of \( T(\mathfrak{A}) \). We are concerned in this section with automorphisms \( S \) of \( \mathfrak{A} \) which determine inner automorphisms \( \Sigma \) of \( T(\mathfrak{A}) \) under (12).

The group \( \mathfrak{g} \) of all inner automorphisms of \( T(\mathfrak{A}) \) is an invariant subgroup of the automorphism group of \( T(\mathfrak{A}) \). If \( \mathfrak{g}_T \) is the group of automorphisms of \( T(\mathfrak{A}) \) described in Theorem 3, then the intersection \( \mathfrak{g}_T \cap \mathfrak{g} \) is an invariant subgroup of \( \mathfrak{g}_T \). But then there is an invariant subgroup \( \mathfrak{g} \) of the automorphism group \( \mathfrak{g} \) of \( \mathfrak{A} \) such that \( \mathfrak{g} \cong \mathfrak{g}_T \cap \mathfrak{g} \) under the correspondence \( S \mapsto \Sigma \) of (12). The elements of \( \mathfrak{g} \) are characterized as those automorphisms of \( \mathfrak{A} \) which are themselves elements of \( T(\mathfrak{A}) \) by

**Theorem 6.** Let \( \mathfrak{A} \) be a non-associative algebra over \( \mathfrak{g} \) with unity element 1 and automorphism \( S \) determining an automorphism \( \Sigma \) of \( T(\mathfrak{A}) \) by (12). Then \( \Sigma \) is inner if and only if \( S \) is in \( T(\mathfrak{A}) \).

If \( S \) is in \( T(\mathfrak{A}) \), then \( T \mapsto T \Sigma = S^{-1}TS \) is an inner automorphism of \( T(\mathfrak{A}) \). Conversely, if \( \Sigma \) is inner, there exists a nonsingular element \( K \) of \( T(\mathfrak{A}) \) such that \( T \Sigma = K^{-1}TK \) for all \( T \) in \( T(\mathfrak{A}) \). In particular, \( R_{xS} = R_x \Sigma = K^{-1}R_xK \). Let \( 1K = k \) so that \( xSL_k = k \cdot xS = kR_{xS} = 1KK^{-1}R_xK = xK \) for all \( x \) in \( \mathfrak{A} \), \( SL_k = K \). Since \( S \) and \( K \) are nonsingular, \( L_k^{-1} \) exists. Moreover, \( L_k^{-1} \) is in \( T(\mathfrak{A}) \), and \( S = KL_k^{-1} \) is in \( T(\mathfrak{A}) \).

Perhaps it should be pointed out that Theorem 6 yields nothing in the case of central simple algebras (that is, algebras which are simple for all scalar extensions). For although it is true that, if \( \mathfrak{A} \) is central simple, then \( T(\mathfrak{A}) \) is also and—by a well known theorem concerning associative algebras—every automorphism \( \Sigma \) of \( T(\mathfrak{A}) \) is inner, so that Theorem 6 implies that every automorphism \( S \) of \( \mathfrak{A} \) is in \( T(\mathfrak{A}) \), it is also true [1, §8] that in this case \( T(\mathfrak{A}) = (\mathfrak{g})_n \), the algebra
of all linear transformations on $\mathfrak{A}$. Of course it is vacuous then to say that $S$ is in $T(\mathfrak{A})$.

5. The right ideals $\mathfrak{R}_T, \mathfrak{R}_R, \mathfrak{R}_L$. We now make a more thorough analysis of the right ideals $\mathfrak{R}_T, \mathfrak{R}_R, \mathfrak{R}_L$ of $T(\mathfrak{A}), E(R(\mathfrak{A})), E(L(\mathfrak{A}))$, respectively, and arrive in particular at criteria for the (right, left) simplicity of an algebra $\mathfrak{A}$ with unity quantity.

**Theorem 7.** An algebra $\mathfrak{A}$ with unity quantity is both commutative and associative if and only if $\mathfrak{R}_T = 0$.

For $\mathfrak{R}_T = 0$ implies that $L_x - R_x = R_x R_y - R_{xy} = 0$ for all $x, y$ in $\mathfrak{A}$. That is,

$$\begin{align*}
R_x &= L_x, \\
R_x R_y &= R_{xy},
\end{align*}$$

$\mathfrak{A}$ is both commutative and associative. Conversely, if (30) holds for all $x, y$ in $\mathfrak{A}$, then $T$ in $T(\mathfrak{A})$ has the form $T = f(R_x, L_x, R_y, \ldots) = g(R_x, R_y, \ldots) = R_{g(x, y, \ldots)}$. Then $1T = 0$ implies $g(x, y, \ldots) = 0$ or $T = 0$. Hence $\mathfrak{R}_T = 0$.

The center $\mathfrak{Z}$ of $\mathfrak{A}$ consists of all elements $c$ in $\mathfrak{A}$ such that

$$\begin{align*}
xc &= cx, \\
c(xy) &= (cx)y = x(cy),
\end{align*}$$

or equivalently

$$\begin{align*}
cL_x &= cR_x, \\
cR_{xy} &= cR_x R_y = cR_y L_x
\end{align*}$$

for all $x, y$ in $\mathfrak{A}$.

**Theorem 8.** An element $c$ is in the center $\mathfrak{Z}$ of an algebra $\mathfrak{A}$ with unity quantity if and only if $c\mathfrak{R}_T = 0$.

Certainly $L_x - R_x, R_{xy} - R_x R_y, R_{xy} - R_y L_x$ are in $\mathfrak{R}_T$ for all $x, y$ in $\mathfrak{A}$. Hence if $c\mathfrak{R}_T = 0$, it follows that $c(L_x - R_x) = c(R_{xy} - R_x R_y) = c(R_{xy} - R_y L_x) = 0$ or (32) holds, $c$ is in the center of $\mathfrak{A}$. Conversely, if $c$ is in the center of $\mathfrak{A}$, and if we write $T$ in $T(\mathfrak{A})$ as in (8), it is seen by repeated application of (32) that $cT = cf(R_x, L_x, R_y, \ldots) = cR_{g(x, y, \ldots)}$, where the non-associative polynomial $g(x, y, \ldots) = 1f(R_x, L_x, R_y, \ldots) = 1T$. But if $T$ is in $\mathfrak{R}_T$, then $1T = 0$ so that $g(x, y, \ldots) = 0$ and $cT = 0$, $c\mathfrak{R}_T = 0$.

An algebra $\mathfrak{A}$, which is not the zero algebra of order 1, is called simple (right simple, left simple) in case the only ideals (right ideals, left ideals) of $\mathfrak{A}$ are 0 and $\mathfrak{A}$.

**Theorem 9.** A non-associative algebra $\mathfrak{A}$ with unity quantity is right simple if and only if $\mathfrak{R}_R$ is a maximal proper right ideal of $E(R(\mathfrak{A}))$.

If $\mathfrak{R}_R$ is a maximal proper right ideal of $E(R(\mathfrak{A}))$, then the only
right ideal of $E(R(\mathfrak{A}))$ containing $\mathfrak{M}_R$ properly is $E(R(\mathfrak{A}))$ itself. We assume that $\mathfrak{A}$ is not right simple, so that $\mathfrak{A}$ has a right ideal $\mathfrak{Q} \neq 0$, $\mathfrak{A}$. Let $\mathfrak{B}$ be the linear set $\mathfrak{B} = R(\mathfrak{Q}, \mathfrak{A}) + \mathfrak{M}_R$. Then $P$ in $\mathfrak{B}$ has the form $P = R_a + N$, $q$ in $\mathfrak{Q}$, $N$ in $\mathfrak{M}_R$, and any element $T$ of $E(R(\mathfrak{A}))$ may be written as $T = R_t + N_1$, $t$ in $\mathfrak{A}$, $N_1$ in $\mathfrak{M}_R$, so that $PT = (R_q + N)(R_t + N_1) = R_q R_t + R_q N_1 + NT = R_q t + (R_q R_t - R_q t) + R_q N_1 + NT$. Now $R_q N_1 = R_a + N_2$ for $a$ in $\mathfrak{A}$, $N_2$ in $\mathfrak{M}_R$, and $1 R_q N_1 = 1 R_a + 1 N_2$ or $a = q N_1$. Since $N_1 = f(R_s, R_y, \ldots)$ while $\mathfrak{Q}$ is a right ideal of $\mathfrak{A}$, it follows that $a = q N_1 = g f(R_s, R_y, \ldots)$ is in $\mathfrak{Q}$. Hence $PT = R_q t + q N_1 + (R_q R_t - R_q t) + N_2 + NT$ is in $\mathfrak{B}$ since $q t + q N_1$ is in $\mathfrak{Q}$ while $R_q R_t - R_q t + N_2 + NT$ is in $\mathfrak{M}_R$. Hence $\mathfrak{B}$ is a right ideal of $E(R(\mathfrak{A}))$ containing $\mathfrak{M}_R$. Since $\mathfrak{Q} \neq 0$, $\mathfrak{A}$, it follows that $R(\mathfrak{Q}, \mathfrak{A})$, being of the same dimension over $\mathfrak{B}$ as $\mathfrak{Q}$, is neither $0$ nor $R(\mathfrak{A})$, and then $\mathfrak{B} \neq \mathfrak{M}_R$, $E(R(\mathfrak{A}))$, a contradiction. Hence $\mathfrak{A}$ is right simple.

Conversely, let $\mathfrak{B}$ be any proper right ideal of $E(R(\mathfrak{A}))$ which contains $\mathfrak{M}_R$. Consider the set $\mathfrak{Q}$ of residue classes $[P]$ modulo $\mathfrak{M}_R$ for $P$ in $\mathfrak{B}$. Then $\mathfrak{Q}$ is a linear subset of $E(R(\mathfrak{A}))/\mathfrak{M}_R \cong \mathfrak{A}$. Moreover, if $[P]$ is any element of $\mathfrak{Q}$, we write $P = R_p + N$ for $p$ in $\mathfrak{A}$, $N$ in $\mathfrak{M}_R$. Let $[R_t]$ be any element of $E(R(\mathfrak{A}))/\mathfrak{M}_R$. Then $P R_t = R_p R_t + N R_t = P_1$ in $\mathfrak{B}$ since $\mathfrak{B}$ is a right ideal of $E(R(\mathfrak{A}))$. Then

$$[P] [R_t] = [R_p] [R_t] = [R_p R_t] = [P_1]$$

in $\mathfrak{Q}$ by (19), and $\mathfrak{Q}$ is a right ideal of $E(R(\mathfrak{A}))/\mathfrak{M}_R \cong \mathfrak{A}$. If $\mathfrak{A}$ is right simple, then either $\mathfrak{Q} = [0]$ or $\mathfrak{Q} = E(R(\mathfrak{A}))/\mathfrak{M}_R$. In the latter case, $\mathfrak{Q}$ contains $[I]$, $\mathfrak{B}$ contains $I + N_1$ for some $N_1$ in $\mathfrak{M}_R$. Since $\mathfrak{B}$ also contains $N_1$, it follows that $I$ is in $\mathfrak{B}$, whence $\mathfrak{B} = E(R(\mathfrak{A}))$, a contradiction. Hence $\mathfrak{Q} = [0]$, $\mathfrak{B} = \mathfrak{M}_R$, and $\mathfrak{M}_R$ is a maximal proper right ideal of $E(R(\mathfrak{A}))$.

An exactly symmetrical argument, involving left multiplications instead of right multiplications, suffices to prove

**Theorem 10.** A non-associative algebra $\mathfrak{A}$ with unity quantity is left simple if and only if $\mathfrak{M}_L$ is a maximal proper right ideal of $E(L(\mathfrak{A}))$.

Only obvious variations on the proof above are required in the proof of

**Theorem 11.** A non-associative algebra $\mathfrak{A}$ with unity quantity is simple if and only if $\mathfrak{M}_T$ is a maximal proper right ideal of $T(\mathfrak{A})$.

For example, to prove the converse part of the theorem, we let $\mathfrak{B}$ be any proper right ideal of $T(\mathfrak{A})$ which contains $\mathfrak{M}_T$, and let $\mathfrak{Q}$ be the linear space of residue classes $[P]$ modulo $\mathfrak{M}_T$ for $P$ in $\mathfrak{B}$. We may write $P = R_p + N = L_p + N_0$ for $N$, $N_0$ in $\mathfrak{M}_T$, and let $[R_t] = [L_t]$ be
any element of $T(\mathfrak{A})-\mathfrak{N}_T$. Then we have (33) as before, where now the quantities involved are residue classes of $T(\mathfrak{A})$ modulo $\mathfrak{N}_T$, but also we have $PL_t = L_p L_t + N_p L_t = P_2$ in $\mathfrak{B}$ so that $[L_t][P] = [L_t][L_p] = [P_2] = [P_2]$ in $\mathfrak{O}$ by (20), and $\mathfrak{O}$ is an ideal of $T(\mathfrak{A})-\mathfrak{N}_T \subseteq \mathfrak{A}$. The remainder of the proof is as before.

We conclude with an analysis of the structure of the right ideal $\mathfrak{N}_T$ of $T(\mathfrak{A})$ in case $\mathfrak{A}$ of order $n$ over $\mathfrak{B}$ (with unity quantity) is simple. In this case $T(\mathfrak{A}) = (\mathfrak{B})_s$ where the center $\mathfrak{B}$ of $\mathfrak{A}$ is a field of degree $t$ over $\mathfrak{F}$, and $n = st$ (see [1, §§8, 19]).

**Theorem 12.** Let $\mathfrak{A}$ be a simple non-associative algebra of order $n = st$ over $\mathfrak{B}$ with unity quantity and with center $\mathfrak{B}$ of degree $t$ over $\mathfrak{F}$. Then $\mathfrak{N}_T = \mathfrak{R} + (\mathfrak{B})_{s-1}$, where the radical $\mathfrak{R}$ of $\mathfrak{N}_T$ has order $(s-1)$ over $\mathfrak{B}$ and the semi-simple component of $\mathfrak{N}_T$ is the total matric algebra $(\mathfrak{B})_{s-1}$ of degree $(s-1)$ over $\mathfrak{B}$.

For $\mathfrak{A}$ is central simple over $\mathfrak{B}$. Let $(1, u_2, \cdots, u_s)$ be a fixed basis of $\mathfrak{A}$ over $\mathfrak{B}$. Then, since $T(\mathfrak{A}) = (\mathfrak{B})_s$, it follows from Theorem 8 that $\mathfrak{N}_T$ (over $\mathfrak{B}$) consists of all $s$-by-$s$ matrices with first row zero. But the structure of this algebra of matrices, with principal idempotent

$E = \begin{pmatrix} 0 & 0 \\ 0 & I_{s-1} \end{pmatrix},$

is easily determined. Its radical $\mathfrak{R}$ consists of all matrices (with elements in $\mathfrak{B}$) of the form

$\begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix}$

where $U$ is any $(s-1)$-by-$1$ matrix. Its semi-simple component consists of all matrices (with elements in $\mathfrak{B}$) of the form

$\begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix}$

where $V$ is any $(s-1)$-rowed square matrix. This is a total matric algebra $(\mathfrak{B})_{s-1}$.

**References**