ON THE TOPOLOGICAL PRODUCT OF PARACOMPACT SPACES

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A space S is defined by Jean Dieudonné[1] to be *paracompact* provided every covering of S by open sets has a neighborhood-finite refinement[2] which covers it. Dieudonné proves that every metric separable space is paracompact, that every paracompact Hausdorff space is normal, and that if S_1 is compact and S_2 is paracompact, then $S_1 \times S_2$ is paracompact. He leaves open the question as to whether the topological product of two paracompact spaces is paracompact. It is the purpose of this note to answer this question in the negative. The example given provides also a simple instance of a normal Hausdorff space S such that $S \times S$ is not normal.

The space S. The points of the space S are the non-negative real numbers, and a neighborhood of a point is any semi-open interval of the form $a \leqq x < b$ which contains it. It is easily seen that S is a regular Hausdorff space which is separable but not perfectly separable.

To show that S is paracompact, let α be any covering of S by open sets. It is evident that there is a refinement β of α whose elements are neighborhoods. Let E be the set of all points e of S which satisfy the relation $a < e < b$ for no neighborhood $a \leqq x < b$ of β. The set E is closed with respect to the natural topology of the real numbers. (For if z is a point in the complement of E, there is a neighborhood $a \leqq x < b$ of β such that $a < z < b$, and since the open interval $a < x < b$ contains no point of E, it follows that z is not a limit point of E.) The complement of E, therefore, is the sum of a countable number of disjoint open intervals $I_1, I_2, \cdots$. For each n the left end point of I_n is a point of E, and each point of E is the left end point of some interval I_n. Hence if I_n is the interval $e_n < x < f_n$, it follows that $E = \sum e_n$ and hence that $S = \sum (I_n + e_n)$. For each n there exists a neighborhood $e_n \leqq x < q_n$ of β; denote it by D_n. Let γ_n be the collection of all open intervals $a < x < b$ such that $a \leqq x < b$ belongs to β and is a subset of $e_n \leqq x < f_n$. Then γ_n covers $q_n \leqq x < f_n$. Since under the natural topology of the real

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[1] *Une generalisation des espaces compacts*, J. Math. Pures Appl. (9) vol. 23 (1944) pp. 65–76.

[2] The covering ω is a *refinement* of a covering α if every element of ω is a subset of some element of α; ω is *neighborhood-finite* if every point of the space has a neighborhood which intersects only a finite number of elements of ω.

numbers $q_n \leqq x < f_n$ is paracompact, it follows that there is a neighborhood-finite refinement δ_n of γ_n which covers it. Let ϵ_n be the collection of intersections of elements of δ_n with $q_n \leqq x < f_n$. The elements of ϵ_n are open sets in the topology of S, and each of them is a subset of an element of α. Hence if η_n is the collection obtained by adding to ϵ_n the open set D_n, η_n is a neighborhood-finite covering of $I_n + e_n$ whose elements are subsets of elements of α; therefore $\eta = \sum \eta_n$ is a neighborhood-finite refinement of α which covers S.

Hence S is paracompact and therefore normal.

The space $S \times S$. The points of $S \times S$ are, of course, sensed pairs (x, y) of non-negative real numbers, and neighborhoods are semi-open rectangles of the form $a \leqq x < b$, $c \leqq y < d$.

To show that $S \times S$ is not paracompact it will be sufficient to show that it is not normal. To this end let H be the set of all points (x, y) of $S \times S$ such that $x + y = 1$ and $[(x-1)^2 + y^2]^{1/2}$ is rational, and let K be the set of all points (x, y) of $S \times S$ such that $x + y = 1$ and $[(x-1)^2 + y^2]^{1/2}$ is irrational. It is readily seen that H and K are disjoint closed sets in $S \times S$. Let V be any open set containing K. For each point (k, k') of K there is a neighborhood $N(k, k')$ of the form $k \leqq x < k + d$, $k' \leqq y < k' + d$ which lies in V. For each positive integer n let K_n be the set of all points (k, k') of K such that the diameter of $N(k, k')$ is greater than $1/n$. Then $K = \sum K_n$, and since K is of the second category with respect to the natural topology of the line interval $x + y = 1$, it follows that there exists an integer m such that K_m is dense in an interval $(s, 1-s)$, $(t, 1-t)$ of $x + y = 1$. It follows easily that every point of the rectangle R bounded by $x + y = 1$, $x + y = 1 + 1/m$, $x - y = 2s - 1$, $x - y = 2t - 1$ belongs to some $N(k, k')$. Hence R is a subset of V. But every point of H in the interval $(s, 1-s)$, $(t, 1-t)$ of $x + y = 1$ is a limit point of R and hence of V. It follows that H is intersected by the closure of every open set containing K.

Hence $S \times S$ is not normal and therefore not paracompact.

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