SOME GENERALIZED HYPERGEOMETRIC POLYNOMIALS

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1. Introduction. We shall obtain some basic formal properties of the hypergeometric polynomials

$$(1)\qquad \begin{aligned} f_n(a_i;\, b_j;\, x) &\equiv f_n(a_1, a_2, \cdots, a_p;\, b_1, b_2, \cdots, b_q;\, x) \\ &\equiv {}_{p+2}F_{q+2}\left[\begin{matrix} -n, & n+1, & a_1, \cdots, a_p; \\ 1/2, & 1, & b_1, \cdots, b_q; \end{matrix}\; x\right] \end{aligned}$$

(n a non-negative integer) in an attempt to unify and to extend the study of certain sets of polynomials which have attracted considerable attention. Some special cases of the $f_n(a_i;\, b_j;\, x)$ are:[1]

(a) $f_n(1/2;\, —;\, x) = P_n(1-2x)$ (Legendre).
(b) $f_n(1;\, —;\, x) = [n!/(1/2)_n]P_n^{(-1/2,1/2)}(1-2x)$ (Jacobi).
(c) $f_n(1,\, 1/2;\, b;\, x) = [n!/(b)_n]P_n^{(b-1,1-b)}(1-2x)$ (Jacobi).
(d) $f_n(1/2,\, \zeta;\, p;\, v) = H_n(\zeta,\, p,\, v)$ [**12**].
(e) $f_n[1/2,\, (1+z)/2;\, 1;\, 1] = F_n(z)$ [**3**].
(f) $f_n(1/2;\, 1;\, t) = Z_n(t)$ [**4**].
(g) $f_n[1/2,\, (z+m+1)/2;\, m+1;\, 1] = F_n^m(z)$ [**8**].

2. A generating function. Let $G(y)$ be analytic at $y=0$,

$$G(y) = \sum_{n=0}^{\infty} c_n y^n,$$

and define $f_n(x)$ by the relation

$$(2)\qquad \frac{1}{1-w}\, G\left[\frac{-4xw}{(1-w)^2}\right] = \sum_{n=0}^{\infty} f_n(x) w^n.$$

If w is sufficiently small, the left member of (2) may be expanded in an absolutely convergent double series and rearranged so as to give a convergent power series in w. Let that be done. Then it is easily shown that

$$(3)\qquad f_n(x) = \sum_{r=0}^{n} \frac{(-n)_r (n+1)_r c_r x^r}{(1/2)_r r!}$$

in which $(a)_r = a(a+1)\cdots(a+r-1);\ (a)_0 = 1$.

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[1] A dash indicates the absence of parameters. A number in brackets relates to the references.

Thus, if $G(y)$ is a generalized hypergeometric function,[2]

$$G(y) = {}_pF_q(a_1, a_2, \cdots, a_p; b_1, b_2, \cdots, b_q; y), \tag{4}$$

then the $f_n(x)$ defined by (2) are precisely the polynomials $f_n(a_i; b_j; x)$ given by (1). This is a special case of a known summation formula given by T. W. Chaundy [5, p. 62]. It is necessary to note that in using the generating function we implicitly demand that the parameters a_i and b_j be independent of n.

Starting with (1) in its expanded form,

$$f_n(a_i; b_j; x) = \sum_{r=0}^{n} \frac{(-n)_r(n+1)_r(a_1)_r \cdots (a_p)_r x^r}{(1/2)_r(b_1)_r \cdots (b_q)_r r!r!}, \tag{7}$$

we may, by replacing r by $n-r$ and reversing the order of summation, write

$$\begin{aligned} f_n(a_i; b_j; x) &= \frac{(-4x)^n(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n n!} \\ &\quad {}_{q+3}F_{p+1}\left[\begin{matrix} -n, -n, 1/2-n, 1-b_1-n, \cdots, 1-b_q-n; \\ -2n, \qquad 1-a_1-n, \cdots, 1-a_p-n; \end{matrix} (-1)^{q-p+1}/x\right]. \end{aligned} \tag{6}$$

3. **A differential recurrence formula.** Let us return to (2) without the restriction that $G(y)$ be of hypergeometric type. Two equations each involving $G'(y)$ may be found by separate differentiations of (2) with respect to x and to w. Then $G'(y)$ may be eliminated and

$$\theta[f_n(x) + f_{n-1}(x)] = n[f_n(x) - f_{n-1}(x)], \tag{7}$$

in which θ is the operator xd/dx.

Direct application of (7) to (d) and (f) of §1 yields the possibly new relations $v[H_n'(\zeta, p, v) + H_{n-1}'(\zeta, p, v)] = n[H_n(\zeta, p, v) - H_{n-1}(\zeta, p, v)]$ (in which primes denote differentiation with respect to v) and

$$t[Z_n'(t) + Z_{n-1}'(t)] = n[Z_n(t) - Z_{n-1}(t)].$$

4. **Pure recurrence relations.**[3] Let $p \leqq q+1$, the parameters a_i and b_j be independent of n, no a_i be equal to any b_j, one, or one-half and let no b_j be equal to a nonpositive integer. Then, for $n \geqq q+4$, $f_n(a_i; b_j; x)$ satisfies a pure linear recurrence relation of exactly $q+4$ terms of the following type:

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[2] Sufficient conditions for the convergence of the series for $G(y)$ in this form for $p \leqq q+1$ are given in [2] and will not be restated here.

[3] The proofs and details of the material in §§4 and 5 can be found in my dissertation at the University of Michigan, 1945.

$$f_n + (A_1 + A_2x)f_{n-1} + (A_3 + A_4x)f_{n-2} + \cdots$$
$$(8) \qquad + (A_{2k-1} + A_{2k}x)f_{n-k} + \cdots$$
$$+ (A_{2q+3} + A_{2q+4}x)f_{n-q-2} + A_{2q+5}f_{n-q-3} = 0,$$

in which the numerators of the rational functions A_{2k} are functions of n and a_i only. If we change the hypothesis slightly so as to permit a_1 to equal either one or one-half, we obtain a linear recurrence relation of $q+3$ terms. If we allow a_1 to equal one and a_2 to equal one-half, there exists a linear recurrence relation of $q+2$ terms.

As an example, Bateman's $Z_n(t)$ [4] has the following recurrence relation:

$$n^2(2n-3)Z_n(t) - (2n-1)[3n^2 - 6n + 2 - 2(2n-3)t]Z_{n-1}(t)$$
$$(9) \qquad + (2n-3)[3n^2 - 6n + 2 + 2(2n-1)t]Z_{n-2}(t)$$
$$- (2n-1)(n-2)^2 Z_{n-3}(t) = 0,$$

which we have not been able to find elsewhere.

Now let $p > q+1$ but retain the remaining conditions in the opening sentence of this section. Then, for $n \geqq p+3, f_n(a_i; b_j; x)$ satisfies a pure linear recurrence relation of exactly $p+3$ terms and of the same type as (8) above.

5. **Contiguous polynomial relations.** For the sake of simplicity, when discussing contiguous polynomials we use the shortened notation $f_n(a_1+)$ to signify $f_n(a_1+1, \cdots, a_p; b_j; x)$ while $f_n(a_m-)$, $f_n(b_m+)$, and $f_n(b_m-)$ have similar meanings. Using Rainville's results [11] it can be shown that the $f_n(a_i; b_j; x)$ have the following canonical set of $2p+q-2$ contiguous polynomial relations:

For $p \leqq q+1$,

$$(10) \quad (a_1 - a_m)f_n = a_1 f_n(a_1+) - a_m f_n(a_m+); \qquad m = 2, 3, \cdots, p,$$

$$(11) \quad (a_1 - b_m + 1)f_n = a_1 f_n(a_1+) - (b_m - 1)f_n(b_m-); m = 1, 2, \cdots, q.$$

For $p < q$, (10) and (11) with

$$4(a_1 - a_m)(a_1 + a_m - 1)f_n$$
$$= a_1(2a_1 - 1)f_n(a_1+) - a_m(2a_m - 1)f_n(a_m+)$$
$$(12) \qquad + (a_1 - 1)(2a_1 - 1)f_n(a_1-) - (a_m - 1)(2a_m - 1)f_n(a_m-)$$
$$- 2(a_1 - a_m)x \sum_{j=1}^{q} T_{j;1,m} f_n(b_j+); \qquad m = 2, 3, \cdots, p,$$

in which

$$T_{j;k,m} = \frac{-(n+b_j)(n+1-b_j)\prod_{s=1,(k,m)}^{p}(a_s-b_j)}{b_j\prod_{s=1,(j)}^{q}(b_s-b_j)},$$

in which $\prod_{s=1,(k,m)}^{p} c_s$ indicates the product of p factors, from which c_k and c_m have been deleted.

For $p=q$, (10) and (11) with

$$\begin{aligned}(13)\qquad & 2(a_1-a_m)[2(a_1+a_m-1)+x]f_n \\ & = a_1(2a_1-1)f_n(a_1+)-a_m(2a_m-1)f_n(a_m+) \\ & \quad + (a_1-1)(2a_1-1)f_n(a_1-)-(a_m-1)(2a_m-1)f_n(a_m-) \\ & \quad - 2(a_1-a_m)x\sum_{j=1}^{q}T_{j;1,m}f_n(b_j+); \qquad m=2,3,\cdots,p.\end{aligned}$$

For $p=q+1$, (10) and (11) with

$$\begin{aligned}(14)\qquad & (a_1-a_m)[4(a_1+a_m-1)-(4a_1+4a_m-2C+2D-3)x]f_n \\ & = a_1(2a_1-1)(1-x)f_n(a_1+)-a_m(2a_m-1)(1-x)f_n(a_m+) \\ & \quad + (a_1-1)(2a_1-1)f_n(a_1-)-(a_m-1)(2a_m-1)f_n(a_m-) \\ & \quad - 2(a_1-a_m)x\sum_{j=1}^{q}T_{j;1,m}f_n(b_j+); \qquad m=2,3,\cdots,p,\end{aligned}$$

in which

$$C=\sum_{s=1}^{p}a_s \quad\text{and}\quad D=\sum_{s=1}^{q}b_s.$$

If we admit $f_{n-1}(a_i;b_j;x)$ and polynomials contiguous to it, we have another contiguous polynomial relation, namely,

$$\begin{aligned}(15)\qquad & (n+a_1)f_n-(n-b_1+1)f_{n-1}-a_1f_n(a_1+) \\ & \qquad - (b_1-1)f_{n-1}(b_1-)=0.\end{aligned}$$

By an extension of the method used by Rainville for the derivation of his formula (30) in [11], it is comparatively easy to obtain a similar set of contiguous function relations applicable to any terminating generalized hypergeometric function in which the number of numerator parameters exceeds the number of those in the denominator by two or more.

6. **Integral relations.** Using standard methods[4] we obtain the integral relations

$$(16)\qquad f_n(a_i;\, b_j;\, x) = \frac{i}{2\pi^{1/2}}\int_{\infty}^{(0+)} (-\,y)^{-1/2}e^{-y}f_n(1/2,\, a_i;\, b_j;\, -\,x/y)dy,$$

$$(17)\qquad f_n(a_i;\, b_j;\, x) = \frac{1}{\pi^{1/2}}\int_0^{\infty} y^{-1/2}e^{-y}f_n(a_i;\, 1/2,\, b_j;\, xy)dy,$$

and

$$(18)\qquad \begin{aligned} f_n(a_i;\, b_j;\, x) &= \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1 - a_1)} \\ &\quad\cdot \int_0^1 y^{a_1-1}(1 - y)^{b_1-a_1-1}f_n(a_2, \cdots, a_p;\, b_2, \cdots, b_q;\, xy)dy, \end{aligned}$$

in which $R(b_1) > R(a_1) > 0$.

Some special cases of interest are:

$$(19)\qquad f_n(—;\, —;\, x) = \frac{i}{2\pi^{1/2}}\int_{\infty}^{(0+)} (-\,y)^{-1/2}e^{-y}P_n[(2x + y)/y]dy.$$

$$(20)\qquad P_n(1 - 2x) = \frac{1}{\pi^{1/2}}\int_0^{\infty} y^{-1/2}e^{-y}f_n(—;\, —;\, xy)dy.$$

$$(21)\qquad H_n(\zeta,\, p,\, v) = \frac{1}{\pi^{1/2}}\int_0^{\infty} y^{-1/2}e^{-y}f_n(\zeta;\, p;\, vy)dy.$$

$$(22)\qquad f_n(\zeta;\, p;\, v) = \frac{i}{2\pi^{1/2}}\int_{\infty}^{(0+)} (-\,y)^{-1/2}e^{-y}H_n(\zeta,\, p,\, -\,v/y)dy.$$

7. **An algebraic relation.** Let us return to the generating function (2). After substituting $1/s$ for w, multiplying both sides of the resulting equation by $1/s$, and taking the inverse Laplace transform of both sides, we have

$$(23)\qquad L^{-1}\left[\frac{1}{s-1}\, G\left(\frac{-\,4xs}{(s-1)^2}\right)\right] = \sum_{n=0}^{\infty} f_n(x)\,\frac{t^n}{n!}\,\cdot$$

Next consider the polynomials $h_n(x)$ defined by

$$(24)\qquad G(2xz - z^2) = \sum_{n=0}^{\infty} h_n(x)z^n/n!,$$

[4] The methods of this section were suggested by those used by Rice in **[12]** for obtaining relations for $H_n(\zeta,\, p,\, v)$.

where $G(y)$ is the same analytic function which appears in (2). Replacing z by $(-2x)/(s-1)$, we obtain

$$(25)\qquad L^{-1}\left[\frac{1}{s-1}G\left(\frac{-4x^2s}{(s-1)^2}\right)\right] = e^t\sum_{n=0}^{\infty} h_n(x)\frac{(-2xt)^n}{(n!)^2}.$$

From (23) and (25) it follows that

$$(26)\qquad f_n(x^2) = \sum_{k=0}^{\infty} C_{n,k}(-2x)^k h_k(x)/k!.$$

In a compact symbolic notation[5] (26) appears as

$$(26)'\qquad f_n(x^2) \doteqdot L_n(2xh(x)),$$

in which $L_n(x)$ is the Laguerre polynomial.

If the function $G(y)$ in (24) is taken to be ${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; y)$, then the resulting $h_n(x)$ are the generalized hypergeometric polynomials

$$(27)\qquad h_n(a_i; b_j; x) = n!\sum_{k=0}^{[n/2]} \frac{(a_1)_{n-k}\cdots(a_p)_{n-k}(2x)^{n-2k}(-1)^k}{(b_1)_{n-k}\cdots(b_q)_{n-k}(n-2k)!k!}.$$

These polynomials are generalizations of the Hermite and the ultraspherical polynomials,

$$h_n(-;-;x) = H_n(x), \qquad h_n(a;-;x) = n!P_n^{(a)}(x).$$

Hence our f_n without parameters is related by (26) to the ordinary Laguerre and Hermite polynomials by

$$f_n(-;-;x^2) \doteqdot L_n(2xH(x)).$$

8. **Bateman's $Z_n(t)$ and the Laguerre polynomials.** From (17) above it follows at once that

$$(28)\qquad f_n(1/2;1;t) = \frac{1}{\pi^{1/2}}\int_0^{\infty} y^{-1/2}e^{-y}f_n(-;1;ty)dy.$$

Ramanujan's formula[6] [6] which gives the product of two ${}_1F_1$'s as an ${}_2F_3$ and which was proved by C. T. Preece [9] is useful here. It yields at once

$$f_n(-;1;x^2/4) = L_n(x)L_n(-x).$$

But Bateman's $Z_n(t)$ is given by

[5] Rainville [**10**, p. 244].
[6] Bailey [**2**, p. 97].

$$Z_n(t) = f_n(1/2; 1; t).$$

With these results it is easy to write (28) in the interesting form

$$Z_n(t^2) = \frac{1}{\pi^{1/2}} \int_0^\infty e^{-\alpha^2/4} L_n(\alpha t) L_n(-\alpha t) d\alpha.$$

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