an even entire function of order one and minimal type. A counter example shows that the word "minimal" cannot be replaced by "normal." (Received June 27, 1947.)

Applied Mathematics

333. H. E. Salzer: *Checking and interpolation of functions tabulated at certain irregular logarithmic intervals.*

For functions that are usually represented upon semi-logarithmic graph paper, that is, which behave as polynomials in $\log x$, the problem of checking or interpolation when the x's are in geometric progression is quite simple due to the uniform interval in $\log x$. But in practice functions are often given at some or all of the points 1, 2, 5, 10, 20, 50, 100, 200, 500, 1000 (same as .001, .002, .005, and so on, or .01, .02, and so on). In the present paper coefficients are given which facilitate: (I) checking of such functions when given at some of the more frequently occurring combinations of those points, by obtaining the last divided difference; (II) Lagrangian interpolation according to a generalization of the scheme recently given by W. J. Taylor, Journal of Research, National Bureau of Standards, vol. 35 (1945) pp. 151–155, RP 1667. (Received July 16, 1947.)

334. H. E. Salzer: *Coefficients for expressing the first twenty-four powers in terms of the Legendre polynomials.*

Exact values of the coefficients of $P_m(x)$, the mth Legendre polynomial, in the expression for x^n as a series of Legendre polynomials are tabulated for $n=0, 1, 2, \cdots, 24$. Previous tables due to Byerly or Hobson do not extend beyond $n=8$, and are inadequate for many needs. These coefficients will be useful in approximating a polynomial of high degree (denoted by $f(x)$ after normalization to the interval $[-1, 1]$) by a polynomial of lower degree which will be best in a well known least square sense; that is, for a preassigned r, they will be used to obtain the polynomial $q_r(x)$ of degree not greater than r which minimizes $\int_{-1}^{1}[f(x)-q_r(x)]^2dx$. (Received July 3, 1947.)

335. H. E. Salzer: *Complex interpolation over a square grid, based upon five, six, and seven points.*

Lagrangian coefficients are tabulated for complex interpolation of an analytic function of $Z=x+iy$, which is given over a square grid in the Z-plane. The formulas employed here are based upon the values of the function at five, six, or seven points which are chosen so as to be as close together as possible, at the sacrifice of possible symmetry. (This is a continuation of the tables contained in the article by A. N. Lowan and H. E. Salzer, *Coefficients for interpolation within a square grid in the complex plane*, Journal of Mathematics and Physics vol. 23 (1944), which gives the coefficients for the 3- and 4-point cases.) Denoting the reference point in the lower left-hand corner by Z_0, and h the length of the grid, so that $Z=Z_0+Ph$, where $P=p+iq$, the approximating n-point formulas are of the well known form $\sum L_j^{(n)}(P)f(Z_j)$. Exact values of the coefficients $L_j^{(n)}(P)$ are given for $p=0, .1, .2, \cdots, 1.0$ and $q=0, .1, .2, \cdots, 1.0$. A method for inverse interpolation is indicated, based upon the coefficients of P^m in $L^{(n)}(P)$. (Received July 16, 1947.)

336. H. E. Salzer: *Further remarks on the approximation of numbers as sums of reciprocals.*

The present work consists of three main parts. (1) Comparison of R-expansions with simple continued fractions for rational numbers a/b leads to the analogue of the Euclidean algorithm, with a multiple of the g.c.d. of a and b in place of the g.c.d. One practical result is that for a/b, as a rule, fewer partial fractions are required in the R-expansion than in the s.c.f. (2) Proof that if p/q is an approximation to x obtained by the R-expansion, then the remainder $(x-p/q)<1/q$. (3) Proof that if p/q is an approximation to x obtained by the $\overline{R}$-expansion, then $|x-p/q|<1/2q$ except when $x=3/4$, when the $\leqq$ relation may hold. Both theorems are best possible ones. It is shown that when in an R- or $\overline{R}$-expansion the denominator of the $(n+1)$th partial fraction is about k times the minimum value that could arise in an R- or $\overline{R}$-expansion, then not only the closeness of the nth, but of all ensuing approximations p/q will be about $1/kq$ or $1/2kq$ respectively. A critical bibliography is provided, which reviews all work by other authors on R- or $\overline{R}$-expansions. (Received June 18, 1947.)

337. H. E. Salzer: *Polynomials of best approximation in an infinite interval.* Preliminary report.

Chebyshev polynomials $C_n(x)$ are useful for approximating polynomials of high degree in a finite interval $[a, b]$, by polynomials of much lower degree, because of the property that of all polynomials with leading coefficient 1, $C_n(x)$ has the least value of the greatest deviation from 0 in the interval $[-1, 1]$. To approximate functions of the form $e^{-x}p(x)$ and $e^{-x^2}q(x)$, where $p(x)$, $q(x)$ are polynomials, over $[0, \infty]$ and $[-\infty, \infty]$ respectively, it is useful to know: (I) polynomials $P_n(x)$, degree n, leading coefficient 1, such that the greatest absolute value of $e^{-x}P_n(x)$ differs least from 0 in $[0, \infty]$; (II) polynomials $Q_n(x)$, degree n, leading coefficient 1, such that the greatest absolute value of $e^{-x^2}Q_n(x)$ differs least from 0 in $[-\infty, \infty]$. $P_n(x)\equiv x^n+a_{n-1}x^{n-1}+\cdots+a_0$ satisfies $2n$ transcendental equations $P_n'(x_i)=P_n(x_i)$, $e^{-x_i}P_n(x_i)=(-1)^ia_0$, $i=1, \cdots, n$, where x_i are the abscissae of the extrema of $e^{-x}P_n(x)$. $Q_n(x)\equiv x^n+b_{n-1}x^{n-1}+\cdots+b_0$ satisfies $2n+1$ equations $Q_n'(x_i)=2x_iQ_n(x_i)$, $e^{-x_i^2}Q_n(x_i)=(-1)^{i-1}e^{-x_1^2}Q_n(x_1)$, $i=1, \cdots, n, n+1$. When suitably normalized, $P_n(x)$ and $Q_n(x)$ are characterized by being tangent alternately to $\pm e^x$ and $\pm e^{x^2}$ respectively. $Q_n(x)\equiv P_{n/2}(x^2)$ for n even, and is an odd function for n odd. (Received July 11, 1947.)

Geometry

338. W. R. Utz: *The properties of geodesics on certain n-dimensional manifolds.* Preliminary report.

Let S denote the interior of an $(n-1)$-dimensional unit sphere in Euclidean n-space and let G denote a properly discontinuous group of homeomorphisms of S onto itself that preserve the hyperbolic metric $\int(dx_idx_i)^{1/2}/(1-x_ix_i)$. The action of this group is discussed in a manner similar to that of Poincaré (*Théorie des groupes Fuchsiens*, Acta Math. vol. 1 (1882) pp. 1–62) for the case $n=2$. By the identification of congruent points of S under G an n-dimensional manifold, Σ, is secured. An investigation of the geodesics on Σ leads to results concerning geodesics and hyperbolic lines of the same type, asymptotic geodesics and limit geodesics analogous to certain results of Morse (*A fundamental class of geodesics on any closed surface of genus greater than one*, Trans. Amer. Math. Soc. vol. 26 (1924) pp. 25–60)) in the case of certain two-dimensional manifolds. (Received May 24, 1947.)