Logic and Foundations

339. David Nelson: *Constructible falsity.*

A definition for constructible truth of number theoretic statements is presented which entails the truth of a statement of the form "Not for all x, $A(x)$" only in case there is an effective method of constructing a natural number n such that "$A(n)$" is false. The definition is similar to that of Kleene (Journal of Symbolic Logic vol. 10 (1945) pp. 109–124) for realizability. Using results of this paper and Nelson (Trans. Amer. Math. Soc. vol. 61 (1947) pp. 307–368) a simply consistent system of number theory satisfying the truth definition is constructed. While some classically acceptable principles which are intuitionistically invalid are reinstated, the system becomes inconsistent upon the adjunction of an axiom schema representing the principle of contradiction. The system contains the systems of classical and intuitionistic number theory on suitable reinterpretation of the logical symbols of those systems. (Received July 28, 1947.)

340. Ira Rosenbaum: *A simple process for obtaining a formula satisfying the nth q-ary truth table of m-valued logic.*

The number, n, of any q-ary truth table of m-valued logic is determinable—by the methods of Bull. Amer. Math. Soc. Abstract 53-5-265—from the relation $n-1=\sum_{j=0}^{c-1} W_{c-j}\cdot m^{c-1-j}$ where $c=m^q$ and $W_i=$ the ith truth-value of the given table minus 1. Divide the summands above into m groups of m^{q-1} summands, and reduce all exponents to the remainders obtained on division of these exponents by m, adding 1 to each subsum obtained. Let the resulting sums be denoted by n_i, $i=1, 2, \cdots, m$. Then the following relation is provable: $F_n^{q(m)}(p_1, \cdots, p_q) \equiv \prod_{i=1}^{m} \{ i(p_q) \supset F_{n_{m-i+1}}^{q-1(m)}(p_1, \cdots, p_{q-1}) \}$ where the left member of the equivalence denotes the nth q-ary truth-function of m-valued logic, $i(p_q)$ denotes a singular function of p_q which has the value 1 when p_q has the value i and the value m otherwise. This relation yields a formula—defined in terms of &, $\supset$, singular functions and $(q-1)$-ary functions—which satisfies the nth q-ary truth table. Recursion on q finally yields a formula with the same property—built only from &, $\supset$, and singular functions—which is, for q or $m \geqq 3$, more easily obtained and comprehensible than conjunctive or disjunctive normal forms. (Received July 22, 1947.)

Statistics and Probability

341. H. D. Brunk: *The strong law of large numbers.*

Sufficient conditions for the strong law of large numbers for sequences of independent random variables involving moments of arbitrary even order are obtained using a method based on Kolmogoroff's. These are given for the strong law in a generalized form analogous to Feller's generalized form of the weak law (see Bull. Amer. Math. Soc. vol. 51 (1945) p. 827). Similar sufficient conditions are also given for sequences of independent random variables for which the moments do not exist. (Received May 12, 1947.)

342. J. L. Doob: *Asymptotic properties of Markoff transition probabilities.*

Let $P^{(n)}(x, A)$ be the probability of a transition from a point x into a set A (Markoff process). It is supposed that there is a self-reproducing distribution, that is, a distribution $\Phi(A)$ satisfying $\int P^{(i)}(x, A)\Phi(dG_x) = \Phi(A)$. Various theorems are proved, generalizing results of Doeblin, Kryloff and Bogoliouboff, and Yosida and Kakutani on the asymptotic properties of $P^{(n)}(x, A)$ for large n. For example if the singular component $P_1(x, A, n)$ of $P^{(n)}(x, A)$ measure with respect to $\Phi(A)$ measure goes to 0 when $n \to \infty$ for all fixed x, the x space can be decomposed into finitely or denumerably many disjunct sets $N, A_1, A_2, \cdots$ such that $\Phi(N) = 0$ and (*) $P^{(n)}(x, A) \to Q(x, A)$ (Cesàro convergence) for all x; here $Q(x, A)$ is a probability measure in A for fixed x, and $Q(x, A) = Q_j(A)$ is independent of x for x in $A \subset A_j$. Each ergodic set A_j can be further subdivided into a finite number d_j of sets through which the system governed by the given transition probabilities runs cyclically; if $d_j = 1$ the limit in (*) is to be taken in the ordinary sense. Previously used hypotheses are equivalent to supposing that $P_1(x, A, n) \to 0$ uniformly in x; (*) is then also uniform in x, and there are at most finitely many A_j's. (Received May 3, 1947.)

343. J. L. Doob: *Renewal theory from the point of view of the theory of probability.*

The renewal problem is considered as a probability problem, leading as usual to the integral equation (*) $U(t) = F(t) + \int_0^t U(t-s)dF(s)$ where $F(s)$ is the probability of death before age s and $U(t)$ is the expected number of deaths in the interval $(0, t)$. The properties of $U(t)$ satisfying (*) are deduced from a probability analysis of the stochastic process involved. For example if $\Delta = \int_0^\infty s dF(s)$, Feller's theorem that $U(t)/t \to 1/\Delta$ is deduced as a consequence of the law of large numbers. Using the theory of Markoff processes it is shown that if $\Delta < \infty$ and if some convolution of $F(s)$ is not a singular function, $\lim_{t\to\infty} [U(t+h) - U(t)] = h/\Delta$ for all h. (Received May 3, 1947.)

344. T. E. Harris: *Some theorems on the Bernoullian multiplicative process.*

A single entity may have j descendents with probability p_j, $j = 0, 1, 2, \cdots$. Each "first generation" entity has then the same procreative probabilities, and so on. Let $f(s) = p_0 + p_1 s + \cdots$. If z_n is the number of entities in the nth generation, it is known that $P(z_n = j)$ is given by the coefficient of s^j in the nth iterate $f[f \cdots (f)] = f_n(s)$. Let $Ez_1 = x$, $1 < x < \infty$. Conditions are given insuring that as $n \to \infty$ the cumulative distribution of the variate z_n/x^n approaches a limit function which is absolutely continuous except for a possible single jump. Let $g(u)$ be the corresponding frequency function. If $f(s)$ is a polynomial of degree k, let $q = \log_x k/(\log_x k - 1)$. Otherwise $q = 1$. Then $g(u) \cdot \exp\{u^{q+\epsilon}\}$ is [is not] summable $(0, \infty)$ according as ϵ is negative [positive]. Behavior of $g(u)$ near $u = 0$ is also considered. Special cases are considered where $g(u) = \text{constant} \cdot u^{1/m-1}e^{-u/m}$, m a positive integer. Maximum likelihood estimates for the parameters $p_0, p_1, \cdots$ and x are obtained as functions of n successive values $z_1, z_2, \cdots, z_n$. Consistency in a certain sense is proved. A specialized method is given for finding the moment-generating function of the variate N, the smallest value of n such that $z_n = 0$. (Received July 21, 1947.)

345. A. M. Mark: *Some probability limit theorems.*

Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed random variables, each having mean 0 and standard deviation 1 and let $S_k = X_1 + X_2 + \cdots + X_k$. It is shown that the limiting distributions of $n^{-1/2} \min (S_{N+1}, S_{N+2}, \cdots, S_n)$ and $n^{-2}\sum_{j=1}^{n} P(j/n)S_j^2$ exist and are independent of the distribution of the X's when

$N=[\alpha n]$, $0<\alpha<1$, and $p(t)$ is a suitably restricted function defined over $0\leqq t\leqq 1$. A similar limit theorem is proved for the distribution of $n^{-2}\sum_{j=1}^{n}S_{nj}^2$ where $S_{nj}=X_{n1}+X_{n2}+\cdots+X_{nj}$ and $X_{n1}, X_{n2}, \cdots, X_{nn}$ are independent, identically distributed random variables each having mean μ_n and standard deviation 1, if $\lim_{n\to\infty} n^{1/2}\mu_n$ is finite. In each case the limiting distribution is first shown to be independent of the distribution of the X's and is then calculated by a convenient choice of a particular distribution of the X's. This method was first used by Erdös and Kac (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 292–302). (Received July 28, 1947.)

346. Kenneth May: *Probabilities of certain election results.*

Consider n election districts in each of which m votes are cast for one of two parties, the district votes following independent uniform distributions. The author discusses the probability $P(m, n)$ that either party gain a majority of the districts with a minority of the total vote. The problem is connected with that of enumerating the compositions of a number into n parts $n\geqq 0$, $n\leqq m$, and such that more than $n/2$ parts are greater than $m/2$. Taking m and n odd to avoid ties, results are stated in terms of $\mu=(m+1)/2$ and $\nu=(n+1)/2$. Utilizing the generating function $(1-z^{\mu})^n(1-z)^{-n}$, the probability is found to be $2^{1-n}\mu^{-n}\sum\sum(-1)^sC_{n,\nu+i}C_{n,s}C_{n+\mu\nu-\mu-\nu-(i+s)\mu,n}$ where the summation extends over i and s satisfying $\mu\nu-\nu-\mu-(i+s)\mu\geqq 0$. In particular, $P(m, 3)=(\mu^2-1)/8\mu^2$. For large n, utilizing Bernstein's central limit theorem, there results $P(m, \infty)=1/2-(1/\pi)\arctan\{\mu 3^{1/2}/(\mu^2-1)^{1/2}\}$. For large m, with the assistance of Fourier integrals, it is shown that $P(\infty, n)=2^{1-n}(n!)^{-1}\sum\sum(-1)^sC_{n,s}C_{n,s+r}r^n$ where the summation is taken over r and s satisfying $\nu-1-r-s\geqq 0$. From $P(3, 3)=3/32$, with increasing m and n, the probability rapidly approaches $P(\infty, \infty)=1/6$. (Received May 21, 1947.)

347. H. E. Robbins: *Some remarks on the inequality of Tchebychef.*

Let C_n denote the class of random variables X such that $X=X_1+\cdots+X_n$ where the X_i are independent and equidistributed and $E(X_i)=0$, $E(X_i^2)=1/n$. Let $\phi_n(t)$ denote the upper bound of $P[|X|\geqq t]$, where X runs over the class C_n. It is shown that $\phi_n(t)<t^{-2}$ for any $n>1$ and sufficiently large t. Thus Tchebychef's inequality can be improved for random variables in C_n, even though, as t becomes infinite, $\phi_n(t)\sim t^{-2}$. (Received July 28, 1947.)

348. Abraham Wald: *On the distribution of the maximum of successive cumulative sums of independent but not identically distributed chance variables.*

For each positive integer N, let $X_{N1}, \cdots, X_{NN}$ be a set of independently distributed chance variables. Put $S_{Ni}=X_{N1}+\cdots+X_{Ni}$ and $M_N=\max(S_{N1}, \cdots, S_{NN})$. It is shown that under some weak restrictions the limiting distribution of $M_N/N^{1/2}$ depends only on the first two moments of X_{Ni}. The exact distribution of M_N is obtained when X_{Ni} can take only the values $+1$ and -1. Lower and upper limits for M_N are given which yield particularly simple lower and upper limits for the probability distribution of M_N when the X's are symmetrically distributed around zero. (Received June 26, 1947.)

349. J. E. Walsh: *Loss of information in t-tests with unbalanced samples.*

Consider two normal populations $N(a_1, \sigma_1^2)$ and $N(a_2, \sigma_2^2)$, where σ_1/σ_2 (ratio of the standard deviations) has a known value C. If the equality of the means, $a_1=a_2$, is to be tested by a t-test (one-sided or symmetrical) using n_1 sample values from $N(a_1, \sigma_1^2)$ and n_2 values from $N(a_2, \sigma_2^2)$ ($n_1+n_2=n$, fixed), it is shown that this experiment is most powerful when $n_1/n_2=\sigma_1/\sigma_2$ (integer considerations neglected). The t-tests satisfying this condition are called balanced. Thus information is lost by not using a balanced experiment. A quantitative measure of the information lost by using given values of n_1 and n_2 is determined by the total sample size m ($m_1+m_2=m$) of the balanced t-test (same significance level) having approximately the same power. Then $n-m$ sample values are wasted by using (n_1, n_2) rather than (m_1, m_2), that is, only $100m/n\%$ of the information obtainable per observation is used by (n_1, n_2). A symmetrical t-test with significance level 2α has the same value of m as a one-sided t-test with significance level α. For one-sided t-tests with significance level α: $m \doteq 2^{-1}(B+(B^2-8A)^{1/2})$, where $B=2+A+K_\alpha^2/2$, $A=(C+1)^2[1-K_\alpha^2/2(n-2)] \cdot [C^2/n_1+1/n_2]^{-1}$, and K_α is the standardized normal deviate exceeded with probability α. This approximation to m is valid for $m \geqq 5$ if $\alpha=.05$, $m\geqq 6$ if $\alpha=.025$, $m\geqq 7$ if $\alpha=.01$, $m\geqq 8$ if $\alpha=.005$. (Received July 16, 1947.)

350. J. E. Walsh: *Some significance tests for the median which are valid under very general conditions.* Preliminary report.

Consider n independent values drawn from populations satisfying only: (1) Each population has a unique median. (2) The median has the same value ϕ for each population. (3) Each population is symmetrical. (4) Each population is continuous. (No two of the values are necessarily drawn from the same population.) Significance tests are derived for ϕ. These tests are based on order statistics of certain combinations of order statistics, each combination being either a single order statistic of the n values or one-half the sum of two order statistics. The tests are reasonably efficient if the values represent a sample from a normal population. The significance levels are of the form $r/2^n$ ($r=1, \cdots, 2^n-1$). Each value of r can be obtained for some one-sided test. The major disadvantage of these tests is the limited number of suitable significance levels for small n. This disadvantage is partially eliminated by the development of tests having a specified significance level if the values are a sample from a normal population and a significance level bounded near this value if only (1)–(4) hold. Applications of these tests furnish generalized results for the Behrens-Fisher problem, certain large "sample" cases, quality control, slippage tests, the sign test, and situations where some of the n values are dependent. (Received July 16, 1947.)

TOPOLOGY

351. S. S. Chern: *On the characteristic ring of a differentiable manifold.*

Let M be a differentiable manifold of dimension n, which may be finite or infinite, orientable or non-orientable, and let $H(n, N)$ be the Grassmann manifold of all the linear spaces of dimension n through a fixed point O of a Euclidean space E^{n+N} of dimension $n+N$, $N\geqq n+2$. By imbedding M in E^{n+N} and constructing through O the linear spaces parallel to the tangent linear spaces of M, a mapping of M into $H(n, N)$ is obtained. This mapping induces a ring homomorphism of the cohomology ring of $H(n, N)$ into the cohomology ring of M, whose image is called the characteristic ring $C(M)$ of M. Take as coefficient ring the ring of residue classes mod 2. Formulas