A NOTE ON THE SCHMIDT-REMAK THEOREM

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Let G be a group with operator domain Ω. We shall say that G satisfies the modified maximal condition for Ω-subgroups if the chain $H_1 \subset H_2 \subset \cdots \subset H \neq G$ is finite whenever $H_1, H_2, \cdots, H$ are Ω-subgroups of G.

Let $A_1, A_2, \cdots$ be a countable set of groups. The direct product of $A_1, A_2, \cdots$ will be defined to be the set of elements $(a_1, a_2, \cdots)$ where a_i is an element of A_i for $i=1, 2, \cdots$, and where but a finite number of the a_i are not the identity elements of the groups in which they lie. A product in the group is defined by the usual componentwise composition of two elements. This group will have the symbol $A_1 \times A_2 \times \cdots$.

The following theorem is in a sense a generalization of the Schmidt-Remak theorem.

THEOREM. *Let G be a group with operator domain Ω, and let Ω contain the inner automorphisms of G. Let $G = A_1 \times A_2 \times \cdots$ where each of the Ω-subgroups A_i is directly indecomposable, and each satisfies the minimal condition and the modified maximal condition for Ω-subgroups. Then if $G = B_1 \times B_2 \times \cdots$ is a second direct product decomposition of G into indecomposable factors, the number of factors will be the same as the number of the A_i. Further the A_i may be so rearranged that $A_i \cong B_i$, and for any j*

$$G = B_1 \times B_2 \times \cdots \times B_j \times A_{j+1} \times A_{j+2} \times \cdots .$$

A proof of the theorem can be based on any standard proof of the Schmidt-Remak theorem such as that given by Jacobson[1] or by Zassenhaus[2] with but slight changes in the two fundamental lemmas.

We state the following lemmas for a group G with operator domain Ω, and we assume that for G and Ω:

(1) Ω contains all inner automorphisms of G.

(2) G satisfies the minimal condition and the modified maximal condition for Ω-subgroups.

(3) G is indecomposable into the direct product of Ω-subgroups.

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[1] Nathan Jacobson, *The theory of rings*, Mathematical Surveys, vol. 2, New York, 1943.

[2] H. Zassenhaus, *Lehrbuch der Gruppentheorie*, Leipzig, 1937.

LEMMA 1. *Let α be an Ω-operator of G. If there exists in G an element h not equal to the identity of G such that $h^\alpha = h$, then α is an automorphism of G.*

This lemma follows by the usual arguments. It is only necessary to note that the fixed point h is sufficient to guarantee that the union of the kernels of the operators $\alpha, \alpha^2, \cdots$ is not G, and that the modified maximal condition then yields that this union is the kernel of some α^k.

LEMMA 2. *Let $\alpha_1, \alpha_2, \cdots$ be addible Ω-operators such that if g is an element of G, then there exists an integer $N(g)$ such that $g^{\alpha_i} = e$, the identity element of G, for all $i > N(g)$. If $\alpha = \alpha_1 + \alpha_2 + \cdots$ is an automorphism of G then, for some k, α_k is an automorphism of G.*

Let g be an element of G, $g \neq e$. Let $\beta_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_N$, $\beta_2 = \alpha_{N+1} + \alpha_{N+2} + \cdots$ where $N = N(g)$. Thus $\alpha = \beta_1 + \beta_2$ and $g^{\beta_2} = e$. We may assume that α is the identity operator. Then $g = g^\alpha = g^{\beta_1} g^{\beta_2} = g^{\beta_1}$. The group G and the operator β_1 satisfy the conditions of Lemma 1, and β_1 is an automorphism of G.

Similarly let $\gamma = \alpha_1 + \alpha_2 + \cdots + \alpha_{N-1}$. Then $\beta_1 = \gamma + \alpha_N$. We may assume that β_1 is the identity operator. If α_N is not an automorphism of G, the kernel of α_N must contain an element $h \neq e$, since G satisfies the minimal condition. Again we may show that γ is an automorphism of G. A repetition of this argument establishes the lemma.

By reference to Lemma 2 the cited proofs of the Schmidt-Remak theorem can be made to yield the following: To each B_i there corresponds a group A_{α_i} where α_i is a positive integral subscript such that $\alpha_i = \alpha_k$ implies $i = k$ and A_{α_i} is operator isomorphic with B_i for all i. Further

$$G = B_1 \times B_2 \times \cdots \times B_j \times A_{\beta_1} \times A_{\beta_2} \times \cdots$$

where $\beta_n \neq \alpha_i$ for any n or i, and where the set of integers $\{\alpha_1, \alpha_2, \cdots, \alpha_j, \beta_1, \beta_2, \cdots\}$ is the set of all positive integers. Let A_m contain the element $g \neq e$. Then for some M, g is an element of the group $B_1 \times B_2 \times \cdots \times B_M$, and since

$$(B_1 \times B_2 \times \cdots \times B_M) \cap (A_{\beta_1} \times A_{\beta_2} \times \cdots) = e,$$

$m \neq \beta_k$ for all k. Thus for some i, $1 \leqq i \leqq M$, we have $m = \alpha_i$, and the set of integers $\{\alpha_1, \alpha_2, \cdots\}$ includes all subscripts. There then exists a reordering of these subscripts such that $\alpha_i = i$.

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