

A NOTE ON THE DERIVATIVES OF INTEGRAL FUNCTIONS

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1. **Introduction.** Let $f(z) = \sum_0^\infty a_n z^n$ be an integral function of order ρ and lower order λ , and $M(r) = \max_{|z|=r} |f(z)|$; $M'(r) = \max_{|z|=r} |f'(z)|$. In a recent paper [1]¹ I have proved the following two theorems.

THEOREM A. *If $f(z)$ be any integral function of order ρ then²*

$$(1.1) \quad \limsup_{r \rightarrow \infty} \frac{\log \{rM'(r)/M(r)\}}{\log r} = \rho.$$

THEOREM B. *If $f(z) = \sum a_n z^n$ be an integral function of lower order λ and $a_n \geq 0$ then*

$$\liminf_{r \rightarrow \infty} \frac{\log \{rM'(r)/M(r)\}}{\log r} = \lambda.$$

The condition that the coefficients a_n be real and non-negative is unnecessary. The purpose of this note is to prove the following two theorems and to deduce a number of interesting results.

THEOREM 1. *If $f(z)$ be an integral function of lower order λ ($0 \leq \lambda \leq \infty$) then*

$$(1.2) \quad \liminf_{r \rightarrow \infty} \frac{\log \{rM'(r)/M(r)\}}{\log r} = \lambda.$$

THEOREM 2. *For any integral function $f(z)$ we have*

$$(1.3) \quad \begin{aligned} \liminf_{r \rightarrow \infty} M'(r)/M(r) &\leq \liminf_{r \rightarrow \infty} \nu(r)/r \leq \limsup_{r \rightarrow \infty} \nu(r)/r \\ &\leq \limsup_{r \rightarrow \infty} M'(r)/M(r), \end{aligned}$$

$$(1.4) \quad \begin{aligned} \liminf_{r \rightarrow \infty} M^{(s+1)}(r)/M^{(s)}(r) &\leq \liminf_{r \rightarrow \infty} \nu(r)/r \leq \limsup_{r \rightarrow \infty} \nu(r)/r \\ &\leq \limsup_{r \rightarrow \infty} M^{(s+1)}(r)/M^{(s)}(r) \quad (s = 1, 2, 3, \dots), \end{aligned}$$

where $f^{(s)}(z)$ is the s th derivative of $f(z)$, $M^{(s)}(r) = \max_{|z|=r} |f^{(s)}(z)|$

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¹ Numbers in brackets refer to the bibliography at the end of the paper.

² A glance at the proof [1, pp. 1-2] shows that the result (1.1) holds when ρ is infinite. An alternative proof of Theorem A is to employ Lemma 4 and relation (8) of my paper [1, p. 1].

and $\nu(r)$ is the rank of the maximum term of $f(z)$ for $|z| = r$.

2. Lemmas.

LEMMA 1. *Let*

(2.1) $F(x)$ be a positive nondecreasing function for $x > x_0$,

(2.2)
$$\liminf_{x \rightarrow \infty} F(x)/x = a \quad (0 \leq a < \infty);$$

then, corresponding to each pair of positive numbers b, c satisfying the inequalities

$$a < b; \quad a/b < c < 1$$

there is a sequence x_1, x_2, \dots , tending to infinity such that

$$F(x) < bx \quad (cx_n \leq x \leq x_n).^3$$

PROOF. For there is a sequence x_1, x_2, \dots such that

$$F(x_n) < bcx_n.$$

Hence if $cx_n \leq x \leq x_n$

$$F(x) \leq F(x_n) < bcx_n \leq bx.$$

LEMMA 2. *Let (2.1) hold,*

(2.3)
$$\limsup_{x \rightarrow \infty} F(x)/x = a_1 \quad (0 < a_1 \leq \infty);$$

then, corresponding to each pair of positive numbers b_1, c_1 , satisfying the inequalities

$$b_1 < a_1; \quad 1 < c_1 < a_1/b_1$$

there is a sequence X_1, X_2, \dots tending to infinity such that

$$F(x) > b_1x \quad (X_n \leq x \leq c_1X_n).$$

For there is a sequence $\{X_n\}$ tending to infinity such that

$$F(X_n) > b_1c_1X_n.$$

If $X_n \leq x \leq c_1X_n$

$$F(x) \geq F(X_n) > b_1c_1X_n \geq b_1x.$$

LEMMA 3 [2, p. 22]. *Let (2.1) hold,*

³ It is always possible to choose the sequence $\{x_n\}$ such that two consecutive intervals have no common point. Similarly for the intervals of Lemmas 2, 3, and 4.

$$(2.4) \quad \liminf_{x \rightarrow \infty} \log F(x)/\log x = \alpha \quad (0 \leq \alpha < \infty);$$

then, corresponding to each pair of positive numbers β, γ satisfying the inequalities

$$\alpha < \beta; \quad \alpha/\beta < \gamma < 1$$

there is a sequence y_1, y_2, \dots tending to infinity such that

$$F(x) < x^\beta \quad (y_n^\gamma \leq x \leq y_n).$$

LEMMA 4. Let (2.1) hold,

$$(2.5) \quad \limsup_{x \rightarrow \infty} \log F(x)/\log x = \alpha_1 \quad (0 < \alpha_1 \leq \infty);$$

then, corresponding to each pair of positive numbers β_1, γ_1 , satisfying the inequalities

$$\beta_1 < \alpha_1; \quad 1 < \gamma_1 < \alpha_1/\beta_1$$

there is a sequence Y_1, Y_2, \dots tending to infinity such that⁴

$$F(x) > x^{\beta_1} \quad (Y_n \leq x \leq Y_n^{\gamma_1}).$$

For there is a sequence $\{Y_n\}$ such that

$$\log F(Y_n) > \beta_1 \gamma_1 \log Y_n.$$

If $Y_n \leq x \leq Y_n^{\gamma_1}$,

$$\log F(x) \geq \log F(Y_n) > \beta_1 \gamma_1 \log Y_n = \beta_1 \log Y_n^{\gamma_1} \geq \beta_1 \log x.$$

3. Proof of Theorem 1. Let

$$\theta(r) = \frac{\log \{rM'(r)/M(r)\}}{\log r}.$$

If λ be infinite, we have from the inequality [3]

$$(3.1) \quad M'(r) > (M(r) \log M(r))/(r \log r); \quad r > r_0 = r_0(f)$$

that⁵

$$\liminf_{r \rightarrow \infty} \theta(r) = \infty.$$

We therefore suppose that $0 \leq \lambda < \infty$. From the inequality (3.1) it follows that

$$(3.2) \quad \liminf_{r \rightarrow \infty} \theta(r) \geq \lambda.$$

⁴ We can deduce Lemmas 1 and 2 from Lemmas 3 and 4.

⁵ r_0 and n_0 are not necessarily the same at each occurrence.

If $\nu(r)$ denotes the rank of the maximum term of $f(z)$, then we have [2, p. 21]

$$\liminf_{r \rightarrow \infty} \log \nu(r) / \log r = \lambda.$$

Hence by Lemma 3, corresponding to each pair of positive numbers β, γ satisfying the inequalities $\lambda < \beta; \lambda/\beta < \gamma < 1$, there is a sequence y_1, y_2, \dots tending to infinity such that

$$\nu(r) < r^\beta \quad (y_n^\gamma \leq r \leq y_n).$$

Let E_n denote the set of points r ($y_n^\gamma \leq r \leq y_n$) and $E = E_1 + E_2 + \dots$. Let F denote the set of points r which lie [4, p. 105] outside a set of exceptional segments in which, for $r > R$, the variation of $\log r$ is less than $K\nu(R/k)^{-1/12}$. Since the variation of $\log r$ in E_n is

$$\log y_n - \gamma \log y_n = (1 - \gamma) \log y_n,$$

which tends to infinity with n , there are points in E_n which do not belong to the set of exceptional segments. The set EF therefore contains a sequence e_1, e_2, \dots tending to infinity. At these points [4, p. 105] e_n

$$(3.3) \quad rM'(r) \sim M(r)\nu(r), \quad r = e_n,$$

$$(3.4) \quad rM^{(s+1)}(r) \sim M^{(s)}(r)\nu(r).$$

Hence⁵ for $n > n_0$

$$e_n M'(e_n) / M(e_n) < 2\nu(e_n) < 2e_n^\beta$$

and so $\liminf_{r \rightarrow \infty} \theta(r) \leq \beta$ and since $\beta - \lambda$ can be chosen arbitrarily small we have

$$\liminf_{r \rightarrow \infty} \theta(r) \leq \lambda$$

and so $\liminf_{r \rightarrow \infty} \theta(r) = \lambda$.

4. Proof of Theorem 2. Let $\liminf_{r \rightarrow \infty} \nu(r)/r = a$ and suppose first that $a < \infty$. Then if

$$a < b, \quad a/b < c < 1,$$

we have, by Lemma 1, $\nu(r) < br$ ($cx_n \leq r \leq x_n$). Let E_n denote the set of points r ($cx_n \leq r \leq x_n$) and $E = E_1 + E_2 + \dots$. The variation of $\log r$ in E_n is $\log x_n - \log cx_n = \log 1/c$ which is not less than $K\nu(R/k)^{-1/12}$ if R be large enough. The total variation of $\log r$ in the intervals $\sum_{p=1}^n E_p$ tends to infinity with n . Hence the set EF contains [4, p. 105]

a sequence e'_1, e'_2, \dots tending to infinity. For $r = e'_n$ ($n > n_0$)

$$(4.1) \quad M'(r)/M(r) \sim \nu(r)/r < b,$$

$$(4.2) \quad M^{(s+1)}(r)/M^{(s)}(r) \sim \nu(r)/r < b \quad (s = 1, 2, 3, \dots).$$

Hence

$$\liminf_{r \rightarrow \infty} M'(r)/M(r) \leq b.$$

Since $b - a$ can be chosen arbitrarily small

$$\liminf_{r \rightarrow \infty} M'(r)/M(r) \leq a$$

which certainly holds if $a = \infty$. Also

$$\liminf_{r \rightarrow \infty} M^{(s+1)}(r)/M^{(s)}(r) \leq a \quad (s = 1, 2, \dots).$$

Let $\limsup_{r \rightarrow \infty} \nu(r)/r = a_1$ and suppose $a_1 > 0$. Let $b_1 < a_1, 1 < c_1 < a_1/b_1$. If G denotes the set of points formed by the intervals of Lemma 2 the set GF contains a sequence $g_1, g_2, \dots, g_n, \dots$ tending to infinity. For $r = g_n$ ($n > n_0$)

$$(4.3) \quad M'(r)/M(r) \sim \nu(r)/r > b_1,$$

$$(4.4) \quad M^{(s+1)}(r)/M^{(s)}(r) \sim \nu(r)/r > b_1 \quad (s = 1, 2, \dots).$$

Hence $\limsup_{r \rightarrow \infty} M'(r)/M(r) \geq a_1$,

$$\limsup_{r \rightarrow \infty} M^{(s+1)}(r)/M^{(s)}(r) \geq a_1$$

which hold if $a_1 = 0$. Hence the theorem follows.

5. Applications. We have from (1.1) and (1.2)

$$(5.1) \quad \begin{aligned} \limsup_{r \rightarrow \infty} \frac{\log \nu(r)}{\log r} = \rho &= 1 + \limsup_{r \rightarrow \infty} \frac{\log \{M'(r)/M(r)\}}{\log r} \\ &= 1 + \limsup_{r \rightarrow \infty} \frac{\log \{M^{(s+1)}(r)/M^{(s)}(r)\}}{\log r}, \end{aligned}$$

$$(5.2) \quad \begin{aligned} \liminf_{r \rightarrow \infty} \frac{\log \nu(r)}{\log r} = \lambda &= 1 + \liminf_{r \rightarrow \infty} \frac{\log \{M'(r)/M(r)\}}{\log r} \\ &= 1 + \liminf_{r \rightarrow \infty} \frac{\log \{M^{(s+1)}(r)/M^{(s)}(r)\}}{\log r} \end{aligned} \quad (s = 1, 2, \dots).$$

Let s denote any fixed positive integer and C, C_1 two positive constants.

(5.3) *If $M'(r) \geq CM(r)$ for all $r > r_0$ then either $\lambda > 1$ or $\lambda = 1$ and $\liminf_{r \rightarrow \infty} \nu(r)/r \geq C$.*

From (1.3) we have $\liminf_{r \rightarrow \infty} \nu(r)/r \geq C$. Hence $\lambda \geq 1$.

REMARK (i). This is a best possible result in the sense that there are functions for which $M'(r) \geq CM(r)$ and $\nu(r) \sim Cr$. We may take for instance $f(z) = \exp(Cz)$.

(ii) The converse—if $\liminf_{r \rightarrow \infty} \nu(r)/r \geq C$ then $M'(r) \geq CM(r)$ for all $r > r_0$ —is false. For consider $f(z) = \cosh Cz$. Here $\lambda = 1 = \rho, \nu(r) \sim Cr$.

$$\frac{M'(r)}{M(r)} = C \frac{e^{Cr} - e^{-Cr}}{e^{Cr} + e^{-Cr}} < C \quad \text{for all } r > 0,$$

$$\frac{M''(r)}{M'(r)} = C \frac{e^{Cr} + e^{-Cr}}{e^{Cr} - e^{-Cr}} > C \quad \text{for all } r > 0,$$

and so on.

(5.4) *If $M^{(s+1)}(r) \geq CM^{(s)}(r), r > r_0$, then $\lambda \geq 1$ and $\liminf_{r \rightarrow \infty} \nu(r)/r \geq C$.*

(5.5) *If $M'(r) \leq C_1M(r)$, or if $M^{(s+1)}(r) \leq C_1M^{(s)}(r)$, for all $r > r_0$, then either $\rho < 1$ or $\rho = 1$ and $\limsup_{r \rightarrow \infty} \nu(r)/r \leq C_1$. This follows from Theorem 2. If $f(z)$ is of order 1 then [5, p. 81] it follows that $\limsup_{r \rightarrow \infty} \log M(r)/r \leq C_1$.*

Let $\phi(r)$ be any function, nondecreasing and positive for $r > r_0$, and such that $\log \phi(r)/\log r$ tends to zero as r tends to infinity.

(5.6) *If $M'(r) \geq (1/\phi(r))M(r)$, or if $M^{(s+1)}(r) \geq (1/\phi(r))M^{(s)}(r)$ for a sequence of values of r tending to infinity, then $\rho \geq 1$.*

This follows from (5.1). If this hypothesis holds for all $r > r_0$ then from (5.2) we get $\lambda \geq 1$.

(5.7) *If $M'(r) \leq \phi(r)M(r)$, or if $M^{(s+1)}(r) \leq \phi(r)M^{(s)}(r)$, for a sequence of values of r tending to infinity, then $\lambda \leq 1$.*

This follows from (5.2). If this hypothesis holds for all $r > r_0$ then from (5.1) we get $\rho \leq 1$.

(5.8) *If $1/\phi(r) \leq M'(r)/M(r) \leq \phi(r)$ or if $1/\phi(r) \leq M^{(s+1)}(r)/M^{(s)}(r) \leq \phi(r)$ for all $r > r_0$ then $\lambda = \rho = 1$.*

This follows from (5.6) and (5.7).

(5.9) *If $\rho < 1$ then*

$$M(r) > \phi(r)M'(r) > \phi^2(r)M''(r) > \dots > \{\phi(r)\}^s M^{(s)}(r)$$

for all $r > r_0$.

This follows⁶ from (5.1).

⁶ Cf. (5.6) above.

(5.10) *If $\lambda > 1$ then*

$$M(r) < \frac{1}{\phi(r)} M'(r) < \frac{1}{\phi^2(r)} M''(r) < \dots < \frac{1}{\{\phi(r)\}^s} M^{(s)}(r)$$

for all $r > r_0$

This follows⁷ from (5.2).

(5.11) *If $\lambda = 1$ and $\liminf_{r \rightarrow \infty} \nu(r)/(r \log r) > 1$ then⁸ $M(r) < M'(r) < M''(r) < \dots < M^{(s)}(r)$ for all $r > r_0$. If $\liminf_{r \rightarrow \infty} \nu(r)/(r \log r) = l > 1$ then since*

$$\log M(r) > \int_{r_0}^r \{\nu(t)/t\} dt,$$

$$\liminf_{r \rightarrow \infty} \log M(r)/(r \log r) > 1.$$

Hence for all $r > r_0$

$$M'(r)/M(r) > \log M(r)/(r \log r) > 1,$$

$$M''(r)/M'(r) > \log M'(r)/(r \log r) > \log M(r)/(r \log r) > 1,$$

and so on.

(5.12) *If $\lambda = 1$ and $\liminf_{r \rightarrow \infty} \nu(r)/r < 1$ there is a sequence of numbers r tending to infinity for which*

$$M(r) > M'(r) > \dots > M^{(s)}(r).$$

Let $\liminf_{r \rightarrow \infty} \nu(r)/r = a$ and $a < b < 1$. The result follows from (4.1) and (4.2). This result does not hold if $\liminf_{r \rightarrow \infty} \nu(r)/r \geq 1$. In fact for the function $f(z) = \cosh z$, $\nu(r) \sim r$ and the sequence $\{M^{(n)}(r)\}$ ($n = 0, 1, 2, \dots$) is not monotonic for any $r > 0$.

(5.13) *If $\lambda < 1$, there is a sequence of numbers r tending to infinity for which*

$$M(r) > \phi(r)M'(r) > \phi^2(r)M''(r) > \dots > \{\phi(r)\}^s M^{(s)}(r).$$

This follows from (3.3) and (3.4).

(5.14) *If $\rho = 1$ and $\limsup_{r \rightarrow \infty} \nu(r)/r > 1$, there is a sequence of numbers r tending to infinity for which*

$$M(r) < M'(r) < M''(r) < \dots < M^{(s)}(r).$$

This follows from (4.3) and (4.4). It does not hold if $\limsup_{r \rightarrow \infty} \nu(r)/r \leq 1$.

(5.15) *If $\rho > 1$, there is a sequence of numbers r tending to infinity*

⁷ Cf. (5.7) above.

⁸ S. K. Bose has proved (5.11) with the hypothesis $\lambda > 2$.

for which

$$M(r) < \frac{1}{\phi(r)} M'(r) < \frac{1}{\phi^2(r)} M''(r) < \dots < \frac{1}{\{\phi(r)\}^s} M^{(s)}(r).$$

Let $1 < \beta_1 < \rho$; $1 < \gamma_1 < \rho/\beta_1$; H the set of intervals of Lemma 4. The set FH contains a sequence of numbers r tending to infinity for which the above inequality holds.

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