

GENERALIZATION OF CERTAIN THEOREMS OF G. SZEGÖ ON THE LOCATION OF ZEROS OF POLYNOMIALS

GYULA DE SZ. NAGY

Using the theorem of Grace, the following result was obtained by G. Szegö:¹

Let the polynomial

$$f(z) = z^n + A_1 z^{n-1} + \dots + A_n$$

have no zeros in the circular region $|z| \leq R$. Then the "section"

$$h(z) = f(z) - z^n = A_1 z^{n-1} + A_2 z^{n-2} + \dots + A_n$$

has no zeros in the circular region $|z| \leq R/2$.

In case of an even n the example $f(z) = (z - R)^n$ shows that the circle $|z| \leq R/2$ can not be replaced by a larger concentric circle. But in case n is odd, according to Szegö the polynomial $h(z)$ is different from zero even in the circle $|z| \leq (R/2) \sec(\pi/2n)$.²

This theorem can be generalized as follows:

I. *Let the polynomial*

$$f(z) = z^n + A_1 z^{n-1} + \dots + A_n$$

have no zeros in the circular region $|z - \alpha| \leq R$. Then no polynomial

$$h(z) = f(z) - \epsilon(z - \alpha)^n, \quad |\epsilon| \leq 1,$$

can have any zeros in the circle $|z - \alpha| \leq R/2$.

The example $f(z) = (z - R)^n$, $\alpha = 0$, $\epsilon = 1$, shows that this theorem can not be refined even in the case of an odd n .

Theorem I is a consequence of the following more general theorem:

II. *Let the polynomial*

$$(1) \quad f(z) = (z - a_1)(z - a_2) \dots (z - a_n)$$

have no zeros in the circle $|z - \alpha| \leq R$; and let the polynomial

$$(2) \quad g(z) = (z - b_1)(z - b_2) \dots (z - b_n)$$

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¹ G. Szegö, *Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen*, Math. Zeit. vol. 13 (1922) pp. 28-55.

² Loc. cit. p. 46.

have all its zeros in the circle $|z - \alpha| \leq \rho, \rho < R$. Then the polynomial

$$(3) \quad h(z) = f(z) - \lambda g(z), \quad |\lambda| \leq t^n, 0 \leq t < R/\rho,$$

can have no zero in the circle

$$(4) \quad |z - \alpha| \leq r = \frac{R - \rho t}{1 + t}.$$

In order to prove this we observe first that for any zero ξ of the polynomial $h(z)$

$$\frac{f(\xi)}{g(\xi)} = \prod_{k=1}^n \frac{\xi - a_k}{\xi - b_k} = \lambda.$$

Hence $h(z_0) \neq 0$ in every point z_0 where

$$\left| \frac{f(z_0)}{g(z_0)} \right| = \prod_{k=1}^n \left| \frac{z_0 - a_k}{z_0 - b_k} \right| \neq |\lambda|.$$

Now at every point z_0 of the circular region (4)

$$|z_0 - a_k| \geq |a_k - \alpha| - |z_0 - \alpha| \geq |a_k - \alpha| - r > R - r = (r + \rho)t,$$

$$|z_0 - b_k| \leq |z_0 - \alpha| + |b_k - \alpha| \leq r + \rho,$$

so that

$$\left| \frac{f(z_0)}{g(z_0)} \right| = \prod_{k=1}^n \left| \frac{z_0 - a_k}{z_0 - b_k} \right| > \left(\frac{R - r}{r + \rho} \right)^n = t^n \geq |\lambda|.$$

This concludes the proof of II. In the case $b_1 = b_2 = \dots = b_n = \alpha, |\lambda| = |\epsilon| = 1 (\rho = 0, t = 1)$, we obtain I.

Substituting

$$g(z) = z^n + A_k z^{n-k} = z^{n-k}(z^k + A_k), \quad \alpha = 0, \quad \rho = |A_k|^{1/k}, \quad \lambda = 1$$

in Theorem II we obtain:

III. The polynomial

$$f(z) = z^n + A_k z^{n-k} + A_{k+1} z^{n-k-1} + \dots + A_n$$

has at least one zero in the circle $|z| \leq 2r + |A_k|^{1/k}$ provided the section

$$h(z) = f(z) - z^n - A_k z^{n-k} \equiv A_{k+1} z^{n-k-1} + A_{k+2} z^{n-k-2} + \dots + A_n$$

has at least one zero on the circle $|z| \leq r$.

The proof of the following theorem is similar to that of II:

IV. Let a_1, a_2, \dots, a_n be given points of the complex plane which are all different from α . Let S_k be the half-plane containing α and having as boundary the perpendicular bisector of $\alpha_1 a_k$. Finally let S^* be the common part of the half-planes: S_1, S_2, \dots, S_n . Then no polynomial

$$h(z) \equiv (z - a_1)(z - a_2) \cdots (z - a_n) - \epsilon(z - \alpha)^n, \quad |\epsilon| \leq 1,$$

can have a zero in the interior of the convex domain S^* .

Indeed in any point z_0 in the interior of S^* we have $|z_0 - a_k| > |z_0 - \alpha|$ so that

$$\left| \frac{f(z_0)}{g(z_0)} \right| = \prod_{k=1}^n \left| \frac{z_0 - a_k}{z_0 - \alpha_k} \right| > 1 \geq |\epsilon|.$$

A theorem similar to II holds also if the zeros of the polynomials $f(z)$ and $g(z)$ are in arbitrary circular domains without common points. One of these circular domains is the interior of a circle, the other the exterior or interior of a circle or a half-plane. Corresponding to these cases three theorems can be obtained generalizing also certain theorems of G. Szegő.³

V. Let the zeros of the polynomials $f(z) = (z - a_1) \cdots (z - a_n)$ and $g(z) = (z - b_1)(z - b_2) \cdots (z - b_n)$ be located in the circular regions

$$(5) \quad |z - \alpha| \geq \rho_1 \quad \text{and} \quad |z - \beta| \leq \rho_2,$$

respectively. We assume that these regions have no points in common, that is,

$$(6) \quad \rho_1 - \rho_2 > 0, \quad |\beta - \alpha| < \rho_1 - \rho_2.$$

Then no polynomial

$$(7) \quad h(z) = f(z) - \epsilon g(z), \quad |\epsilon| \leq 1,$$

can have a zero in the interior of the ellipse E with foci at α and β and with the major axis $\rho_1 - \rho_2$.

VI. Let the zeros of the polynomials $f(z)$ and $g(z)$ be located in the circular regions

$$(8) \quad |z - \alpha| \leq \rho_1 \quad \text{and} \quad |z - \beta| \leq \rho_2,$$

respectively, such that these regions have no points in common, that is,

$$(9) \quad |\beta - \alpha| > \rho_1 + \rho_2.$$

Then no polynomial

³ Loc. cit. pp. 47-48, Theorems 13-15.

$$h(z) = f(z) - \epsilon g(z), \quad |\epsilon| = 1,$$

can have a zero in the interior of the hyperbola H with foci at α and β and with the real axis $\rho_1 + \rho_2$.

VII. Let the zeros of the polynomials $f(z)$ and $g(z)$ be located in the circular region $|z - \alpha| \leq \rho$ and in the half-plane S , respectively, such that these regions have no points in common. Let K be a conic section with α as focus and the boundary line L of the half-plane S as the directrix corresponding to α .⁴ Then no polynomial

$$(10) \quad f(z) - \lambda g(z)$$

with

$$(11) \quad |\lambda| \geq t^n = \left(e + \rho \frac{e+1}{\delta} \right)^n$$

can have a zero in the interior of the conic section K where e is the numerical eccentricity of K and δ is the distance of α from the line L .

By the interior of a conic section we mean the set of points from which no tangent can be drawn to the given conic section.

In order to prove V and VI, we denote by z_0 an arbitrary point in the interior of the conic sections E and H , respectively; let

$$|z_0 - \alpha| = r_1, \quad |z_0 - \beta| = r_2.$$

As to the ellipse E we have

$$\begin{aligned} r_1 + r_2 < \rho_1 - \rho_2 \quad \text{OR} \quad \rho_2 + r_2 < \rho_1 - r_1, \\ |z_0 - a_k| \geq |a_k - \alpha| - |z_0 - \alpha| = |a_k - \alpha| - r_1 > \rho_1 - r_1, \\ |z_0 - b_k| \leq |b_k - \beta| + |z_0 - \beta| \leq \rho_2 + r_2. \end{aligned}$$

From this Theorem V follows since

$$\left| \frac{z_0 - a_k}{z_0 - b_k} \right| > \frac{\rho_1 - r_1}{\rho_2 + r_2} > 1 \quad \text{so that} \quad \left| \frac{f(z_0)}{g(z_0)} \right| > 1 \geq |\epsilon|.$$

As to the hyperbola H , we distinguish two cases regarding the position of z_0 , according as z_0 is nearer to β than to α , or conversely. In the first case

$$\begin{aligned} r_1 > r_2, \quad r_1 - r_2 > \rho_1 + \rho_2, \quad \text{hence} \quad r_1 - \rho_1 > r_2 + \rho_2, \\ |z_0 - a_k| \geq |z_0 - \alpha| - |a_k - \alpha| \geq r_1 - \rho_1, \\ |z_0 - b_k| \leq |z_0 - \beta| + |b_k - \beta| \leq r_2 + \rho_2. \end{aligned}$$

⁴ That is, the polar of α .

In the second case

$$\begin{aligned} r_1 < r_2, \quad r_2 - r_1 > \rho_1 + \rho_2, \quad \text{hence} \quad r_2 - \rho_2 > r_1 + \rho_1, \\ |z_0 - a_k| &\leq |z_0 - \alpha| + |a_k - \alpha| \leq r_1 + \rho_1, \\ |z_0 - b_k| &\geq |z_0 - \beta| - |b_k - \beta| \geq r_2 + \rho_2. \end{aligned}$$

Thus we have in the first and second case,

$$\left| \frac{z_0 - a_k}{z_0 - b_k} \right| \geq \frac{r_1 + \rho_1}{r_2 + \rho_2} > 1, \quad \text{and} \quad \left| \frac{z_0 - a_k}{z_0 - b_k} \right| \leq \frac{r_1 + \rho_1}{r_2 - \rho_2} < 1,$$

respectively. This furnishes Theorem VI since we have

$$\left| \frac{f(z_0)}{g(z_0)} \right| > 1 = |\epsilon|, \quad \text{and} \quad \left| \frac{f(z_0)}{g(z_0)} \right| < 1 = |\epsilon|,$$

respectively.

This proof furnishes the following corollary:

VI'. *Let H be the hyperbola defined in Theorem VI. No polynomial $f(z) - \lambda g(z)$ with $|\lambda| \geq 1$ ($|\lambda| \leq 1$) can have any zero inside the branch of H containing the focus α (β).*

Let z_0 be a point in the interior of the conic section K defined in Theorem VII; let r and d be the distance of z_0 from the focus α and the directrix L , respectively. Then

$$\frac{r}{d} < e \quad \text{and} \quad \frac{1}{d} < \frac{1}{d^*} = \frac{e + 1}{\delta};$$

indeed if z^* is the point of K nearest to L and d^* is the distance of z^* from L , we have

$$\frac{|z^* - \alpha|}{d^*} = \frac{\delta - d^*}{d^*} = e.$$

But $|z_0 - a_k| \leq |z_0 - \alpha| + |a_k - \alpha| \leq r + \rho$ and $|z_0 - b_k| \geq d$, so that

$$\begin{aligned} \left| \frac{z_0 - a_k}{z_0 - b_k} \right| &\leq \frac{r + \rho}{d} < e + \frac{\rho}{d} = e + \rho \frac{e + 1}{\delta} = t, \\ \left| \frac{f(z_0)}{g(z_0)} \right| &\leq \left(\frac{r + \rho}{d} \right)^n < t^n = |\lambda|. \end{aligned}$$

This establishes the proof of Theorem VII.

The special cases

$$b_1 = b_2 = \cdots = b_n = 0, \beta = 0, \rho_2 = 0, \epsilon = 1 \quad \text{of V and VI,}$$

and

$$a_1 = a_2 = \cdots = a_n = 0, \alpha = 0, \rho = 0, e = 1, \lambda = 1 \quad \text{of VII,}$$

are due to G. Szegő.

UNIVERSITY OF HUNGARY, SZEGED

SOME REMARKS ON POLYNOMIALS

P. ERDÖS

This note contains some disconnected remarks on polynomials.

Let $f_n(x) = \prod_{i=1}^n (x - x_i)$, $-1 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1$. Denote by $-1 \leq y_1 \leq \cdots \leq y_{n-1} \leq 1$ the roots of $f'_n(x)$. We prove the following theorem.

THEOREM 1. *For all n*

$$(1) \quad |f_n(-1)| + |f_n(+1)| + \sum_{i=1}^{n-1} |f_n(y_i)| \leq 2^n.$$

For $n \geq 3$

$$(2) \quad |f_n(-1)|^{1/2} + |f_n(+1)|^{1/2} + \sum_{i=1}^{n-1} |f_n(y_i)|^{1/2} \leq 2^{n/2}.$$

For $n \geq n_0(k)$

$$(3) \quad |f_n(-1)|^{1/k} + |f_n(+1)|^{1/k} + \sum_{i=1}^{n-1} |f_n(y_i)|^{1/k} \leq 2^{n/k}.$$

REMARK. If $y_i = y_{i+1}$ or $-1 = y_1$, $+1 = y_{n-1}$ the corresponding summands clearly vanish.

Clearly

$$\begin{aligned} |f_n(-1)| &\leq (1 - x_1)2^{n-1}, & |f_n(y_i)| &\leq |y_i - x_{i+1}| 2^{n-1}, \\ |f_n(+1)| &\leq (1 - x_n)2^{n-1}. \end{aligned}$$

Thus

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