

$$b_1 = b_2 = \cdots = b_n = 0, \beta = 0, \rho_2 = 0, \epsilon = 1 \quad \text{of V and VI,}$$

and

$$a_1 = a_2 = \cdots = a_n = 0, \alpha = 0, \rho = 0, e = 1, \lambda = 1 \quad \text{of VII,}$$

are due to G. Szegő.

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SOME REMARKS ON POLYNOMIALS

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This note contains some disconnected remarks on polynomials.

Let $f_n(x) = \prod_{i=1}^n (x - x_i)$, $-1 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1$. Denote by $-1 \leq y_1 \leq \cdots \leq y_{n-1} \leq 1$ the roots of $f'_n(x)$. We prove the following theorem.

THEOREM 1. *For all n*

$$(1) \quad |f_n(-1)| + |f_n(+1)| + \sum_{i=1}^{n-1} |f_n(y_i)| \leq 2^n.$$

For $n \geq 3$

$$(2) \quad |f_n(-1)|^{1/2} + |f_n(+1)|^{1/2} + \sum_{i=1}^{n-1} |f_n(y_i)|^{1/2} \leq 2^{n/2}.$$

For $n \geq n_0(k)$

$$(3) \quad |f_n(-1)|^{1/k} + |f_n(+1)|^{1/k} + \sum_{i=1}^{n-1} |f_n(y_i)|^{1/k} \leq 2^{n/k}.$$

REMARK. If $y_i = y_{i+1}$ or $-1 = y_1$, $+1 = y_{n-1}$ the corresponding summands clearly vanish.

Clearly

$$\begin{aligned} |f_n(-1)| &\leq (1 - x_1)2^{n-1}, & |f_n(y_i)| &\leq |y_i - x_{i+1}| 2^{n-1}, \\ |f_n(+1)| &\leq (1 - x_n)2^{n-1}. \end{aligned}$$

Thus

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$$\begin{aligned}
& |f_n(-1)| + |f_n(+1)| + \sum_{i=1}^{n-1} |f_n(y_i)| \\
& \leq \left((1-x_1) + \sum_{i=1}^{n-1} (y_i - x_{i+1}) + (1-x_n) \right) 2^{n-1} \leq 2^n,
\end{aligned}$$

which proves (1)

We have by the inequality of the geometric and arithmetic mean

$$\begin{aligned}
|f_n(-1)|^{1/2} & \leq \frac{(1-x_1) + (1-x_2)}{2} 2^{n/2-1}, \\
|f_n(y_i)|^{1/2} & \leq \frac{(x_{i+1} - x_i)}{2} 2^{n/2-1}, \\
|f_n(+1)|^{1/2} & \leq \frac{(1-x_n) + (1-x_{n-1})}{2} 2^{n/2-1}.
\end{aligned}$$

Thus we evidently have for $n \geq 3$

$$|f_n(-1)|^{1/2} + |f_n(+1)|^{1/2} + \sum_{i=1}^{n-1} |f_n(y_i)|^{1/2} \leq 2^{n/2},$$

which proves (2).

$f_1(x) = x$ and $f_2(x) = x^2/2 - 1$ shows that (2) is false for $n < 3$. Clearly equality occurs in (1) and (2) only for $\pm(1 \pm X)^n$.

The proof of (3) is more complicated and since the proof does not present any particular interest we are just going to sketch it. Let $f_n(x)$ be the polynomial which maximizes the sum (3). If (3) is not true we must have

$$|f_n(x_0)| = \max_{-1 \leq x \leq 1} |f_n(x)| > \frac{2^n}{n^k}.$$

But then it is easy to see that x_0 does not lie in $(1-\epsilon, -1+\epsilon)$; without loss of generality we can assume that $1-\epsilon < x_0 \leq 1$, and $n+o(n)$ of the x_i are in $(-1+\delta, -1)$. But then a simple computation shows that $f_n(x)$ has no roots in $(1-\epsilon, 1)$ and thus $x_0=1$. (This is clear since if we move any possible root of $f_n(x)$ in $(1-\epsilon, 1)$ to -1 , we clearly increase the sum (3).) By the same argument we obtain by a simple calculation that all the roots of $f_n(x)$ have to be in -1 , which proves (3) and completes the proof of Theorem 1.

At present I can not determine the exact value of $n_0(k)$.

Let $g_n(z) = \prod_{i=1}^n (z - z_i)$, $|z_i| = 1$. Denote by u_1, u_2, \dots the local extremal points of $g_n(z)$, that is, the points where the vector $g'_n(z)$

points either towards the origin or away from it. I conjectured that

$$\sum_i |g_n(u_i)| \leq 2^n.$$

Professor Breusch¹ proved this conjecture for sufficiently large n . The proof is complicated. For small values of n he showed by examples that the result is false.

Let $-1 = x_0 < x_1 \leq \dots \leq x_n = x_{n+1} = 1$. Put $\omega(x) = \prod_{i=1}^n (x - x_i)$, $l_k(x) = \omega(x) / \omega(x_k)(x - x_k)$, the fundamental functions of Lagrange interpolation. The problem of determining the set for which

$$\max_{-1 \leq x \leq 1} \sum_{k=1}^n |l_k(x)|$$

is minimal is still unsolved. It has been conjectured, but never proved, that the minimum is attained for the points for which all the $n + 1$ sums

$$(4) \quad \max_{x_i \leq x \leq x_{i+1}} \sum_{k=1}^n |l_k(x)|, \quad i = 0, 1, \dots, n,$$

are equal. If the x_i are the roots of $T_n(x)$ (the n th Tchebycheff polynomial), then a simple computation shows that the sums (4) all equal

$$\frac{2}{\pi} \log n + O(1).$$

S. Bernstein² proved that for any $-1 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq 1$

$$\max_{-1 \leq x \leq 1} \sum_{k=1}^n |l_k(x)| > (1 + o(1)) \frac{2}{\pi} \log n,$$

and I proved³ that

$$\max_{-1 \leq x \leq 1} \sum_{k=1}^n |l_k(x)| > \frac{2}{\pi} \log n - c \quad (c \text{ absolute constant}).$$

We consider a slightly different problem. We prove the following theorem.

THEOREM 2. *Let $-1 = x_0 \leq x_1 \leq \dots \leq x_n \leq x_{n+1} = 1$. Then for some i*

$$(5) \quad \max_{x_i < x < x_{i+1}} \sum_{k=1}^n |l_k(x)| < n^{1/2}.$$

¹ Oral communication.

² Bull. Acad. Sci. URSS. Sér. Math. (1931) pp. 1025–1050.

³ Unpublished.

REMARK. $n^{1/2}$ in (5) can very likely be improved to $c \log n$. In fact it is likely that

$$\min_{0 \leq x_i \leq n} \max_{x_i \leq x \leq x_{i+1}} \sum_{k=1}^n |l_k(x)|$$

assumes its maximum when all the sums (4) are equal.

If, for some i , $x_i = x_{i+1}$ then (5) is obvious. Assume that $x_i \neq x_{i+1}$, $0 \leq i \leq n+1$. Consider the equation $\sum_{k=1}^n l_k^2(x) = 1$. The number of solutions is not greater than $2n-2$ and x_1, x_2, \dots, x_n are solutions. Thus a simple argument shows that for some i , $1 \leq i \leq n-1$,

$$\sum_{k=1}^n l_k^2(x) < 1 \quad \text{for } x_i < x < x_{i+1}.$$

But then clearly (from Schwartz's inequality)

$$\sum_{k=1}^n |l_k(x)| < n^{1/2} \quad \text{for } x_i \leq x \leq x_{i+1}$$

which proves Theorem 2.

In one of his interesting papers Schur⁴ proves among others the following result: Let $a_0x^n + \dots + a_n$ be a polynomial with integer coefficients, all whose roots are in $(-1, +1)$ and are different. Then for sufficiently large n , $|a_0| > (2^{1/2} - \epsilon)^n$. We prove the stronger theorem:

THEOREM 3. Let $f_n(x) = a_0x^n + \dots + a_n$ be a polynomial with integer coefficients and $f_n(-1) \neq 0$, $f_n(0) \neq 0$, $f_n(+1) \neq 0$. Then, $|a_0| \geq 2^{n/2}$.

We have (x_1, x_2, \dots, x_n are the roots of $f_n(x)$)

$$|f_n(-1)f_n^2(0)f_n(+1)| = \left| a_0^4 \prod_{i=1}^n (1 - x_i^2)x_i^2 \right| \geq 1.$$

But $|(1 - x_i^2)x_i^2| \leq 1/4$. Thus $|a_0| \geq 2^{n/2}$ which completes the proof. $2^n(x-1/2)^n$ shows that $|a_0| \geq 2^{n/2}$ is best possible.

Schur in his proof makes use of the fact that the discriminant of $f_n(x)$ has to be an integer. If we make use of this fact we easily obtain that, for large n , $|a_0| > (2^{1/2} + c)^n$. On p. 390 of his paper Schur constructs a polynomial of degree $2n$ with

$$a_0^{(2n)} = \frac{1}{2(2)^{1/2}} ((1 + 2^{1/2})^{n+1} - (1 - 2^{1/2})^{n+1}).$$

⁴ Math. Zeit. vol. 1 (1918) pp. 377-402, see p. 389-391.

This seems the greatest possible value of $|a_0^{(2n)}|$.

In the same paper (Theorem XIII, pp. 397–398) Schur proves the following theorem: Let a_0 be a given integer, and let $f_n(z) = a_0 z^n + \dots + a_n$ be a polynomial with integer coefficients the roots of which either all have absolute value 1 and are different or all are in the interior of the unit circle (in which case multiple roots are permitted). Denote these roots by z_1, z_2, \dots, z_n . Then

$$(6) \quad \limsup \frac{z_1 + z_2 + \dots + z_n}{n} \leq 1 - \frac{e^{1/2}}{2}.$$

Schur conjectures that the limit (6) is 0, and remarks that if $a_0 = 1$ this follows from a theorem of Kronecker, which asserts that in this case all the z_i are roots of unity. We now prove Schur's conjecture.

THEOREM 4. *Let the z_i be defined as above. Then*

$$\lim \frac{z_1 + z_2 + \dots + z_n}{n} = 0.$$

First we can assume that n tends to infinity (that is, for every n there are only a finite number of equations satisfying the conditions of the theorem). Also if $f(z)$ has all its root in the interior of the unit circle then $z^2 f(z) + z^n f(z^{-1}) = g(z)$ has all its roots on the unit circle and all are different. Also the sum of the roots of $f(z)$ and $g(z)$ are identical (p. 397). Thus it will suffice to consider polynomials having all their roots on the unit circle and distinct.

Therefore the discriminant of D satisfies the inequality

$$(7) \quad 1 \leq D = a_0^{2n-2} \prod_{i < j \leq n} (z_i - z_j)^2.$$

It follows from a result of Pólya (p. 395) that

$$(8) \quad |f(z_1, z_2, \dots, z_n)| = \prod_{i < j \leq n} (z_i - z_j) \leq n^n.$$

Thus for at least one z

$$(9) \quad \prod_{j \neq i} |z_i - z_j| \leq n.$$

To prove Theorem 4 it clearly suffices to show that the z_i are uniformly distributed on the unit circle. Suppose this is not true. Then it follows from a result of Fekete⁵ that there exists a z_0 , $|z_0| = 1$, such that

⁵ Ann. of Math. vol. 41 (1940) pp. 162–173, see pp. 165–166.

$$(10) \quad \prod_{j \neq i} |z_0 - z_j| > (1 + c_1)^n.$$

But then from (9) and (10)

$$|f(z_0, z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)| > \frac{(1 + c_1)^n}{n} f(z_1, z_2, \dots, z_n).$$

If z_1, z_2, \dots, z_n are not uniformly distributed we can continue this process c_2n times, and thus obtain $y_1, y_2, \dots, y_n, |y_i| = 1$, so that

$$(11) \quad |f(y_1, y_2, \dots, y_n)| > \frac{(1 + c_1)^{c_2n^2}}{n^{c_2n}} |f(z_1, z_2, \dots, z_n)|.$$

But from (7) and (11) we obtain

$$|f(y_1, y_2, \dots, y_n)| > (1 + c_3)^{n^2} \frac{1}{a_0^{2n-2}} > n^n$$

which contradicts (8) and completes the proof of Theorem 4.

Szegő⁶ proved the following theorem: Let M be any closed set in the plane. Denote by $\omega_n(M, z_0)$ the maximum of $|f'_n(z_0)|$ for all polynomials $f_n(z)$ of degree n which satisfy $|f_n(z)| \leq 1$ for all z in M . Assume that the transfinite diameter of M is positive. Then

$$\lim \omega_n(M, z_0)^{1/n} < \infty.$$

Fekete⁷ proved that if z_0 is not in M , $\lim \omega_n(M, z_0)^{1/n}$ exists and is finite if the transfinite diameter of M is positive, and is infinite if the transfinite diameter of M is 0.

Assume now that z_0 belongs to M . The following questions remain open: (1) Does $\lim \omega_n(M, z_0)^{1/n}$ exist? (2) Let the transfinite diameter of M be 0. Is $\lim \omega_n(M, z_0)^{1/n} = \infty$?⁷

We are going to answer both questions in the negative. In fact we prove the following theorem.

THEOREM 5. *Let M be the set consisting of 0 and $1/2^k, k=0, 1, 2, \dots$. Then*

$$\omega_n(M, 0) < c^n.$$

Clearly M is closed and countable, thus its transfinite diameter is 0.

LEMMA. *Let a, b, d be three real numbers, $d - b = b - a$. Then if $|f_n(z)| < 1$ for $a < z < b$,*

⁶ Math. Zeit. vol. 23 (1925) pp. 45-61.

⁷ Math. Zeit. vol. 26 (1927) pp. 324-344.

$$f'_n(d) < c_1^n / (b - a).$$

If $a = 0, b = 1$ the lemma follows from Szegő's⁶ result. The general case follows by a linear transformation. As a matter of fact it is well known that $\omega_n(M, d)$ is given in this case by the Tchebychev polynomial belonging to (a, b) .

Now we prove Theorem 5. The equation $f_n^2(z) = 1$ can have at most $2n$ real roots. Thus since $|f(1/2^k)| < 1, k = 0, 1, \dots$, we obtain that, for some $k > n + 1, |f_n(z)| < 1$ for all $1/2^k < z < 1/2^{k+1}$. Thus by the lemma

$$|f'_n(0)| < 2^{n+1} c_1^n < c^n,$$

q.e.d.

THEOREM 6. *Let the set M be defined as follows: $n_1 < n_2 < \dots$ tend to infinity sufficiently fast. M consists of the points 0 and $1/2^u$ where $n_i \leq u \leq 2n_i + 1$. Then $\lim \omega_n(M, 0)^{1/n}$ does not exist. In fact $\limsup \omega_n(M, 0)^{1/n} = \infty, \liminf \omega_n(M, z_0)^{1/n} < \infty$.*

As in Theorem 5 it follows that if $|f(1/2^u)| < 1$ for $n_i \leq u \leq 2n_i + 1, f(x)$ a polynomial of degree n_i , then $\omega_{n_i}(M, 0) < c^{n_i}$. Consider $f(x) = 2^{n_i+1} \prod_{k=1}^{n_i+1} (x - 1/2^k), k = 1, 2, \dots, 2n_i + 1$. The degree of $f(x)$ equals $2n_i + 2$. Also $|f(x)| < 1$ for all x in M , and if n_{i+1} tends to infinity sufficiently fast

$$(f'(0))^{1/2n_i+2} > (2^{n_i+1} / 2^{(2n_i+1)^2})^{1/2n_i+2} \rightarrow \infty$$

q.e.d.

THEOREM 7. *Let $f_n(z)$ be a polynomial of degree n with real coefficients. $|f_n(z)| < 1$ for $-1 \leq z \leq 1$. Then if $|z_0| \geq 1$*

$$|f_n(z_0)| \leq |T_n(z_0)|.$$

Equality holds only for $f_n(z) = \pm T_n(z)$.

In case z_0 is real this result is well known.

We are going to prove the following more general result: Let $|f_n(x_i)| \leq 1$ where $x_1 = -1, x_i, i = 1, 2, \dots, n - 1$, are the roots of $T'_n(x)$ and $x_n = 1$. Then for $|z_0| \geq 1$

$$(12) \quad |f_n(z_0)| \leq |T_n(z_0)|.$$

We have $(l_i(x) = \omega(x) / \omega'(x_i)(x - x_i), \omega(x) = (1 - x^2)T'_n(x))$

$$f_n(z_0) = \sum_{i=0}^n y_i l_i(x_0), \quad |y_i| \leq 1, y_i \text{ real.}$$

We evidently have for complex numbers A and B , $\max(|A+B|, |A-B|) > A$. Thus $|f_n(z)|$ will be maximal if $y_i = \pm 1$. A simple geometric argument shows that the angle between any two of the vectors $(-1)^i l_i(z_0)$ is less than $\pi/2$ (since the interval $(-1, +1)$ subtends from z_0 at an angle not greater than $\pi/2$). But then clearly $|f_n(z_0)|$ is maximal if

$$f_n(z_0) = \pm \sum_{i=0}^n (-1)^i l_i(z_0) = \pm T_n(z_0).$$

Equality clearly occurs only if $f(z) = \pm T(z)$.

COROLLARY. *Let $|f_n(z)| \leq 1$ for $-1 \leq z \leq 1$, also let $f_n(z)$ have real coefficients. Then for $|z| \leq 1$, $|f_n(z)| < |T_n(z)|$.*

If we do not assume that the coefficients of $f_n(z)$ are real it is easy to give examples which show that $|f_n(z)|$ does not have to be less than $|T_n(z)|$. Trivially $|f_n(z)| \leq \sum_{k=0}^n |l_k(z)|$. But in general $\max |f_n(z)| < \sum_{k=0}^n |l_k(z)|$. I can not at present determine $\max |f_n(z)|$ for $|z| \leq 1$.

In the same way we can prove that if $f(z) = a_0 z^n + \dots + a_n$ has real coefficients and $|f(z)| \leq 1$ for $-1 \leq z \leq 1$ then $\sum_{k=0}^n |a_k|$ is maximal for $f(z) = \pm T_n(z)$. Szegő⁸ proved the following stronger result: $|a_{2k}| + |a_{2k+1}|$ is maximal for $f(z) = \pm T_n(z)$.

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⁸ Oral communication.