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ON THE DARBOUX TANGENTS

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1. **Introduction.** In a recent paper [1][1] Abramescu gave a metrical characterization of the cubic curve obtained by equating to zero the terms of the expansion of a surface S at an ordinary point O_1, up to and including the terms of the third order. This cubic curve is rational and its inflexions lie on the three tangents of Darboux through O_1. In this paper we give a projective characterization of such a curve, and hence a new derivation of the tangents of Darboux. By using the method employed in this characterization to the curve of intersection of the tangent plane of the surface at O_1 with S, a simple characterization of the second edge of Green is found. Another application exhibits the correspondence of Moutard. Finally a new interpretation of the reciprocal of the projective normal is given in terms of the conditions of apolarity of a cubic form to a quartic form. The canonical tangent appears in a similar fashion.

Let S be referred to its asymptotic curves, and let the coordinates (x^1, x^2, x^3, x^4) of the generic point O_1 of S be normalized so that they satisfy the system [2] of differential equations

$$
\begin{aligned}
x_{uu} &= \theta_u x_u + \beta x_v + px, \\
x_{vv} &= \gamma x_u + \theta_v x_v + qx, \qquad \theta = \log R.
\end{aligned} \tag{1.1}
$$

The line l_1 joining O_1 to O_4, whose coordinates are x^i_{uv}, is the R-conjugate line, and the line l_2 determined by O_2, O_3, whose respective coordinates are x^i_u, x^i_v, is the R-harmonic line.

If we define the local coordinates (x_1, x_2, x_3, x_4) with respect to

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[1] Numbers in brackets refer to the references cited at the end of the paper.

$O_1O_2O_3O_4$ of a point X by the expression

$$\overset{i}{X} = x_1\overset{i}{x} + x_2\overset{i}{x}_u + x_3\overset{i}{x}_v + x_4\overset{i}{x}_{uv},$$

and local nonhomogeneous coordinates (x, y, z) by $x=x_2/x_1$, $y=x_3/x_1$, $z=x_4/x_1$, the power series expansion [4] of S at O_1 is

$$z = xy - \frac{1}{3}(\beta x^3 + \gamma y^3) + \frac{1}{12}F_4(x, y) + \cdots, \tag{1.2}$$

wherein

$$\begin{aligned} F_4(x, y) = {} & (2\beta\theta_u - \beta_u)x^4 - 4(\beta\theta_v + \beta_v)x^3y - 6\theta_{uv}x^2y^2 \\ & - 4(\gamma\theta_u + \gamma_u)xy^2 + (2\gamma\theta_v - \gamma_v)y^4. \end{aligned} \tag{1.3}$$

2. **Characteristic points of a plane curve.** Let the triangle of reference $O_1O_2O_3$ to which a plane curve C is referred be covariant to the curve or to a surface to which C bears some geometrical relation. Let the homogeneous coordinates of a point with respect to this triangle be (x_1, x_2, x_3), the nonhomogeneous coordinates being defined by the expressions $x=x_2/x_1$, $y=x_3/x_1$. The line $y=0$ being chosen as the tangent to C at O_1, the power series expansion [4] of C at O_1 is

$$y = a_2x^2 + a_3x^3 + a_4x^4 \cdots. \tag{2.1}$$

Consider at $O_3(0, 0, 1)$ the involution whose double lines are O_1O_3, O_2O_3. Corresponding lines of this involution intersect C in points $P_1(x, y)$, $P_2(-x, y')$, $y'=a_2x^2-a_3x^3+a_4x^4- \cdots$. The line P_1P_2 intersects the tangent to C at O_1 in a point whose limit T as P_1 approaches O_1 along C has coordinates

$$x_1 = a_3, \qquad x_2 = -a_2, \qquad x_3 = 0. \tag{2.2}$$

We shall call the point T with coordinates (2.2) *the characteristic point* of the second order of C at O_1 *relative to $O_1O_3O_2$.*

Let $O_2'(\rho, 1, 0)$ be an arbitrary point on the tangent to C at O_1, but distinct from O_1. The transformation from the triangle $O_1O_2O_3$ to $O_1O_2'O_3$ is

$$x = \frac{Ax'}{1+\rho Ax'}, \qquad y = \frac{By'}{1+\rho Ax'}. \tag{2.3}$$

Under the transformation (2.3), the equation of C may be written in the form

$$y' = a_2'x'^2 + a_3'x'^3 + \cdots,$$

wherein

$$a_2' = A^2 a_2/B, \qquad a_3' = A^3(a_3 - \rho a_2)/B.$$

Hence *the characteristic point of C relative to $O_1O_3O_2'$ has coordinates*

$$(2.4) \qquad x_1 = (a_3 - 2\rho a_2), \qquad x_2 = - a_2, \qquad x_3 = 0$$

referred to $O_1O_2O_3$.

More generally let the equation of C have the form

$$y = a_k x^k + a_{k+1}x^{k+1} + \cdots, \qquad k \geqq 2.$$

Consider through O_3 two lines forming with O_1O_3, O_2O_3 the constant cross ratio l, l being one of the kth roots of unity, but $l \neq 1$. These lines intersect C in two points P_1, P_2 determining a line which intersects the tangent to C at O_1 in a point whose limit as P_1 approaches O_1 has coordinates

$$(2.5) \qquad x_1 = a_{k+1}, \qquad x_2 = - a_k, \qquad x_3 = 0.$$

We shall call the point T whose coordinates are (2.5) *the characteristic point of the kth order of C relative to $O_1O_3O_2$.*

3. **The characteristic curve of** S. Let us consider the section C_π of the surface S by a plane π through the R-conjugate line l_1. Let π intersect the R-harmonic line l_2 in O_π. The local coordinates of O_π are of the form $(0, \lambda, \mu, 0)$, and the local coordinates of any point Q_1 on O_1O_π are $(1, \lambda\xi, \mu\xi, 0)$. The equation of C_π referred to $O_1O_\pi O_4$ in nonhomogeneous coordinates (ξ, z) is

$$(3.1) \qquad z = \lambda\mu\xi^2 - \frac{1}{3}(\beta\lambda^3 + \gamma\mu^3)\xi^3 + \frac{1}{12}F_4(\lambda, \mu)\xi^4 + \cdots.$$

From (2.2) the characteristic point T_π of C_π relative to $O_1O_4O_\pi$ has coordinates

$$(3.2) \qquad \xi = 3\lambda\mu/(\beta\lambda^3 + \gamma\mu^3), \qquad z = 0,$$

referred to $O_1O_\pi O_4$, and coordinates

$$(3.3) \qquad x = 3\lambda^2\mu/(\beta\lambda^3 + \gamma\mu^3), \quad y = 3\lambda\mu^2(\beta\lambda^3 + \gamma\mu^3), \quad z = 0$$

referred to $O_1O_2O_3O_4$. *The locus of T_π* as π rotates about l_1 *is the covariant rational cubic curve Γ_3 whose equation is*

$$(3.4) \qquad 3xy - (\beta x^3 + \gamma y^3) = 0, \qquad z = 0.$$

We shall call this cubic *the characteristic curve of S relative to l_1, l_2. The nodal tangents of Γ_3 are of course the asymptotic tangents of S at O_1, and the inflexions lie on the tangents of Darboux. The R-harmonic line* *is the flex-ray of* Γ_3.

From (3.3) it follows that *the only sections of S through the R-conjugate line whose characteristic points relative to* $O_1O_4O_\pi$ *lie on the R-harmonic line are those through the tangents of Darboux.*

Another characterization of the cubic Γ_3 may be found in the following manner. The osculating conic of the section C_π has the equation [4]

$$\begin{aligned}(3.5)\qquad &\lambda^3\mu^3(z - \lambda\mu\xi^2) + \frac{1}{3}\lambda^2\mu^2(\beta\lambda^3 + \gamma\mu^3)\xi z \\ &\qquad + \left[\frac{1}{9}(\beta\lambda^3 + \gamma\mu^3)^2 - \frac{1}{12}F_4(\lambda, \mu)\right] z^2 = 0.\end{aligned}$$

The pole of R-conjugate line with respect to this conic is the point T_π' with coordinates

$$\xi = -3\lambda\mu/(\beta\lambda^3 + \gamma\mu^3), \qquad z = 0.$$

The harmonic conjugate of T_π' with respect to O_1O_π is the point T_π defined by (3.2). Incidentally the locus of T_π' is the cubic Γ_3',

$$3xy + \beta x^3 + \gamma y^3 = 0.$$

The tangents of Darboux are thus again exhibited by means of Γ_3'.

Finally we may readily show that *the polar line of the conic* (3.5) *intersects* O_4O_π *in a point whose locus as* π *varies is a rational curve of order seven which intersects the R-harmonic line at its intersections with the tangents of Darboux.*

4. **The edges of Green.** The expansions [4] of the two branches of the curve of intersection of S at O_1 with its tangent plane are

$$\begin{aligned}(4.1)\qquad y &= \frac{1}{3}\beta x^2 - \frac{1}{12}(2\beta\theta_u - \beta_u)^3 + \cdots, \; z = 0; \\ x &= \frac{1}{3}\gamma y^2 - \frac{1}{12}(2\gamma\theta_v - \gamma_v)^3 + \cdots, \; z = 0.\end{aligned}$$

The characteristic point T_u of the first of (4.1) relative to $O_1O_3O_2$ has coordinates

$$(4.2)\qquad x_1 = \frac{1}{4}\left(2\theta_u - \frac{\beta_u}{\beta}\right), \qquad x_2 = 1, \qquad x_3 = x_4 = 0,$$

and the characteristic point T_v of the second relative to $O_1O_2O_3$ has coordinates

$$(4.3) \qquad x_1 = \frac{1}{4}\left(2\theta_v - \frac{\gamma_v}{\gamma}\right), \qquad x_2 = 0, \qquad x_3 = 1, \qquad x_4 = 0.$$

The line joining the harmonic conjugates of T_u and T_v with respect to O_1O_2 and O_1O_3 respectively is Green's edge of the second kind.

This edge of Green may be characterized in another way. The section of S by the plane through the R-conjugate line and the tangent to the asymptotic curve v=const. has the equation

$$(4.4) \qquad z = -\frac{1}{3}\beta x^3 + \frac{1}{12}(2\beta\theta_u - \beta_u)x^4 + \cdots .$$

The characteristic point of the third order of the curve (4.4) *relative to $O_1O_4O_2$, is found from* (2.5) *to have coordinates given by* (4.2); by interchanging the roles of the asymptotic tangents the point (4.3) is characterized. The second edge of Green is therefore given another characterization.

Consider on the tangent to the section (3.1) C_π of S the point $O_\pi'(\rho, 2\lambda, 2\mu, 0)$. From (2.4) we find readily that the characteristic point T of C_π relative to $O_1O_4O_\pi'$ has coordinates

$$(4.5) \qquad x_1 = \rho\lambda\mu + \frac{1}{3}(\beta\lambda^3 + \gamma\mu^3), \qquad x_2 = \lambda^2\mu, \qquad x_3 = \lambda\mu^2, \qquad x_4 = 0.$$

The point P_π on the tangent to C_π at O_1 which with O_1 separates O_π' and O_π harmonically has coordinates $(\rho, \lambda, \mu, 0)$. *Equations* (4.4) *therefore represent a cubic transformation of P_π into the characteristic point of C_π relative to $O_1O_4O_\pi'$.* The polar plane of the point (4.5) with respect to any quadric of Darboux,

$$x_2x_3 - x_1x_4 + k_4x_4^2 = 0,$$

has coordinates

$$(4.6) \qquad \xi_1 = 0, \quad \xi_2 = \lambda\mu^2, \quad \xi_3 = \lambda^2\mu, \quad \xi_4 = -\rho\lambda\mu - \frac{1}{3}(\beta\lambda^3 + \gamma\mu^3).$$

The correspondence (4.6) *between P_π and the polar plane of the characteristic point of C_π relative to $O_1O_4O_\pi'$ is the correspondence of Moutard* ($k = -1/3$). We have previously [3] given a different derivation of this correspondence.

5. **The projective normal.** The surface S' whose equation is

$$(5.1) \qquad z = xy - \frac{1}{3}(\beta x^3 + \gamma y^3)$$

has a unode at O_4, the plane $O_2O_3O_4$ as uniplane, and has contact of the third order with S at O_1; hence S' is completely determined. The projection on their common tangent plane at O_1 of the curve of intersection of S and S' has a quadruple point at O_1, the quadruple tangents being given by

$$F_4(x, y) = 0 \tag{5.2}$$

where $F_4(x, y)$ is defined by (1.3). The lines (5.2) intersect the R-harmonic line in four points F_i, and the Segre tangents intersect this line in three points S_i. It is easy to verify that *the points S_i are apolar to F_i if and only if the R-harmonic line is the reciprocal of the projective normal.* The projective normal is therefore geometrically determined by reciprocation with respect to the quadrics of Darboux.

Finally let the lines l_1, l_2 be the projective normal and its reciprocal; then it readily follows that the polar of the form $\beta x^3+\gamma y^3$ with respect to $F_4(x, y)$ is

$$\phi x - \psi y \tag{5.3}$$

wherein $\phi=\partial \log (\beta\gamma^2)/\partial u$, $\psi=\partial \log (\beta^2\gamma)/\partial v$. The form (5.3) equated to zero is seen to be *the equation of the canonical tangent.*

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