NONLINEAR NETWORKS. IIb

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This note is concerned with the quasi-linear properties of an n-dimensional transformation

$$(1)\qquad \begin{aligned} y_1 &= S_1(x_1, \cdots, x_n), \\ &\cdots\cdots\cdots\cdots, \\ y_n &= S_n(x_1, \cdots, x_n). \end{aligned}$$

More precisely it is shown by imposing certain conditions A on the functions S_i that the transformation has the property of possessing a unique inverse. In this property, then, the transformation is analogous to a nonsingular linear transformation. The actual conditions A imposed on the functions S_i are so chosen that the transformation (1) shall be a generalization of the equations which define the steady flow of current in electrical networks made up of quasi-linear conductors. It is reasonable to believe, however, that the methods developed here can, with suitable modification, be used to study the quasi-linear properties of types of transformations which have nothing to do with electrical networks.

At least three different methods of attack are available: The first method is to set up a form (analogous to a positive definite quadratic form) such that the equations (1) are the conditions that this form take on its minimum value. This insures the existence of a solution. A second positive definite form involving the differences of two transformations proves the uniqueness. It is well known that for linear transformations this method has the disadvantage of being applicable only for self-adjoint transformations. A similar disadvantage appears in the nonlinear case. This method was exploited in two previous notes: *Nonlinear networks*. I, and *Nonlinear networks*. IIa. (No appeal is made in this note to results obtained in the previous notes.)

The second method, which is the one employed in this note, is to impose conditions A_1 such that the Brouwer fixed point theorem is available. This insures the existence of the inverse transformation. Corresponding to the differences of two transformations, we associate a linear transformation somewhat analogous to a differential transformation. Conditions A_2 are then imposed, which insure that the associated linear transformation satisfies A_1. Hence, the linear trans-

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formation has an inverse. But a linear transformation with an inverse has a unique inverse, so transformation (1) has a unique inverse.

The third method is that of induction on the dimension. For example, suppose that all transformations of dimension $n-1$ have a unique inverse. With y_n fixed, solve the last equation for x_n in terms of $x_1, \cdots, x_n-1$ and substitute in the previous equations. If conditions A are so chosen that they are dominant under this inbreeding, it follows that the resulting $(n-1)$-dimensional transformation is of the same form and, hence, has a unique inverse by the inductive hypothesis. It is to be noted that if the conditions A are to be dominant, they must be neither too strong nor too weak. For example, the so-called Maxwell junction equations discussed in the previous note would not be invariant in form under this inbreeding. The physical significance of this fact is that a network with one or more concealed junctions may not in general be described in terms of Maxwell's equations applied to the unconcealed junctions. However, for networks of linear conductors it is possible to show by virtue of the distributive law that Maxwell's equations are applicable to the unconcealed junctions; engineers speak of this as a star-mesh transformation. Hence, in engineering terminology the inductive method may be spoken of as the star-mesh method. Theorem 6 below states that the star-mesh transformation preserves conditions A. An independent proof may be constructed for Theorem 6 so the inductive method could be applied. While the inductive method is direct, it is too synthetic to be illuminating, so it has not been used.

As a concrete example of the type of transformations to be treated here, consider the system

$$y_1 = (x_1 - x_2)^3 + (x_1 - x_3)^3,\ y_2 = (x_2 - x_3)^3 + (2x_2 - x_1)^3,$$
$$y_3 = (3x_3 - 2x_1 - x_2)^3 + (3x_3 - 2x_2 - x_1)^3.$$

Inspection of this system shows that the conditions of Theorem 1 are satisfied. Hence, choosing any real numbers for y_1, y_2, y_3, there is a unique real solution x_1, x_2, x_3. It is assumed hereafter that all functions and constants are real-valued.

1. Connected transformations. Clearly, it is always possible to express transformation (1) in the form

$$\begin{aligned} y_1 &= P_1(x_1, x_1 - x_2, \cdots, x_1 - x_n), \\ y_2 &= P_2(x_2 - x_1, x_2, \cdots, x_2 - x_n), \\ &\cdots\cdots\cdots\cdots, \\ y_n &= P_n(x_n - x_1, x_n - x_2, \cdots, x_n). \end{aligned} \tag{2}$$

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The set of functions $P_i(t_1, t_2, \cdots, t_n)$, $i=1, 2, \cdots, n$, will be called an n-dimensional *connected foundation* if for $i, j=1, 2, \cdots, n$ and all values of the variables:

(a) *P_i is a continuous function of the n variables.*

(b) *P_{ij} is either an increasing function unbounded at $\pm\infty$, or is constant.*

(c) *There is a sequence of integers $a, b, \cdots, g, h$ (dependent on i) such that in the chain $P_{ia}, P_{ab}, P_{bc}, \cdots, P_{gh}, P_{hh}$, each function is unbounded at $\pm\infty$.*

Here P_{ij} indicates P_i considered as a function of t_j, other variables being held fixed at arbitrary values. If the set of functions P_i forms a connected foundation, then transformation (2) will be called a *connected transformation.* These are the conditions A referred to above.

THEOREM 1. *For any assigned values of $y_1, \cdots, y_n$, a connected transformation has a unique solution $x_1, \cdots, x_n$.*

PROOF. The theorem will be an obvious consequence of the following two lemmas. It is clear, moreover, that the lemmas are actually more general than Theorem 1. However, under the conditions of the lemmas, Theorem 4 is not generally true, and in this respect transformation (2) would behave less like a linear transformation.

LEMMA 1. *Transformation* (2) *has at least one solution if*: (a) *P_i is a continuous function.* (b_1) *P_{ij} is nondecreasing.* (c_1) *For each i there is a chain in which the functions are unbounded at $+\infty$ and for each i there is a chain in which the functions are unbounded at $-\infty$.*

PROOF. Let y be a constant vector and let ϵ be a positive number. The relations $v_1=\epsilon x_1+P_1(x_1, x_1-x_2, \cdots, x_1-x_n)-y_1$, and so on, define a continuous vector field v when the vector x ranges in and on a cube with corners at $(\pm k, \pm k, \cdots, \pm k)$, $k>0$. From condition (b_1) it follows for $x_1=k$ that $v_1 \geqq \epsilon k+P_1(0, 0, \cdots, 0)-y_1$. Likewise for $x_1=-k$ it follows that $v_1 \leqq -\epsilon k+P_1(0, 0, \cdots, 0)-y_1$. Clearly, then, for $k>\max |P_i(0, 0, \cdots, 0)-y_i|\epsilon^{-1}$ it follows that on the surface of the cube the vector v is pointing outside the cube. Brouwer's fixed point theorem states that if a continuous vector points outside a cube on the surface, then there is at least one point inside the cube where the vector vanishes.[1] At this point

$$y_i = P_i(x_i - x_1, \cdots, x_i - x_n) + \epsilon x_i.$$

It will now be shown that the quantities x_i are bounded inde-

[1] S. Eilenberg suggested to the writer the use of Brouwer's theorem instead of a more special procedure.

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pendent of ϵ. For some integer i, $x_i \geqq x_j$, $j=1, 2, \cdots, n$. Consider a chain sequence $i, a, \cdots, g, h$, for which the functions P_{ia}, $P_{ab}, \cdots, P_{hh}$ are unbounded at $+\infty$, and first suppose $x_i, x_a, \cdots, x_h$ are all positive. Then since x_i is positive

$$\begin{aligned} y_i &\geqq P_i(x_i - x_1, \cdots, x_i - x_a, \cdots, x_1 - x_n) \\ &\geqq P(0, \cdots, x_i - x_a, \cdots, 0). \end{aligned}$$

Thus there is a positive constant c_i independent of ϵ such that $x_i - x_a \leqq c_i$. Hence, $x_j - x_a \leqq c_i$ for any j and $x_a - x_j \geqq -c_i$. Therefore

$$\begin{aligned} y_a &\geqq P_a(x_a - x_1, \cdots, x_a - x_b, \cdots, x_a - x_n) \\ &\geqq P_a(-c_i, \cdots, x_a - x_b, \cdots, -c_i). \end{aligned}$$

Thus there is a positive constant b_a such that $x_a - x_b \leqq b_a$ and $x_j - x_b \leqq c_i + b_a = c_a$. This process is continued until finally we have

$$\begin{aligned} y_h &\geqq P_h(x_h - x_1, \cdots, x_h, \cdots, x_h - x_n) \\ &\geqq P_h(-c_g, \cdots, x_h, \cdots, -c_g), \end{aligned}$$

and there is a positive constant c_h such that $x_h \leqq c_h$.

$$\begin{aligned} x_i &= (x_i - x_a) + (x_a - x_b) + \cdots + (x_g - x_h) + x_h, \\ x_i &\leqq \quad c_i \quad + \quad c_a \quad + \cdots + \quad c_g \quad + c_h. \end{aligned}$$

The same inequality holds if some member of the sequence $x_i, \cdots, x_h$ is not positive. Suppose x_e is the first member of this sequence which is not positive, then

$$\begin{aligned} x_i &= (x_i - x_q) + (x_a - x_b) + \cdots + (x_d - x_e) + x_e, \\ x_i &\leqq \quad c_i \quad + \quad c_a \quad + \cdots + \quad c_d. \end{aligned}$$

The constants c depend only on the vector y and the growth of the functions P_i. There are such constants for each integer i so the components of the vector x have a finite upper bound.

A symmetrical argument, using chains unbounded at $-\infty$, shows that the components of x have a finite lower bound.

As ϵ approaches zero, it follows that the vector x has at least one limit point, and since P_i is continuous this proves Lemma 1.

LEMMA 2. *Transformation* (2) *may not have more than one solution if*: (b_1) P_{ij} *is nondecreasing.* (c_2) *For each i there is a chain in which each of the functions is an increasing function.*

PROOF. If the functions P_i are homogeneous linear functions, then transformation (2) becomes

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$$
(3)\qquad \begin{aligned}
y_1 &= \quad p_{11}x_1 \quad + p_{12}(x_1 - x_2) + \cdots + p_{1n}(x_1 - x_n),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
y_n &= p_{n1}(x_1 - x_n) + p_{n2}(x_n - x_2) + \cdots + \quad p_{nn}x_n.
\end{aligned}
$$

Conditions (b) and (b_1) are equivalent and state that the constants p_{ij} are non-negative. Conditions (c), (c_1), and (c_2) are equivalent and state that for each i there is a sequence of integers $a, b, \cdots, h$ such that no member of the chain $p_{ia}, p_{ab}, \cdots, p_{gh}, p_{hh}$ vanishes.

Let x and x' be two vectors with transforms y and y'. Let $\delta x = x' - x$ and $\delta y = y' - y$. Let $t_1 = x_1$, $t_2 = x_1 - x_2, \cdots, t_n = x_1 - x_n$ and let $t_1' = x_1'$, $t_2' = x_1' - x_2', \cdots, t_n' = x_1 - x_n'$. Then $\delta y_1 = P_1(t_1', \cdots, t_n') - P_1(t_1, \cdots, t_n) = [P_1(t_1', t_2', \cdots, t_n') - P_1(t_1, t_2', \cdots, t_n')] + [P_1(t_1, t_2', \cdots, t_n') - P_1(t_1, t_2, \cdots, t_n')] + \cdots + [P_1(t_1, \cdots, t_{n-1}, t_n') - P_1(t_1, t_2, \cdots, t_n)]$. Let $p_{1j} = [P_1(t_1, \cdots, t_j', t_{j+1}', \cdots, t_n') - P(t_1, \cdots, t_j, t_{j+1}', \cdots, t_n')]\,(t_j' - t_j)^{-1}$ if $t_j' - t_j \neq 0$ and $P_{ij} = 1$ if $t_j' - t_j = 0$.

Thus $\delta y_1 = p_{11}\delta x_1 + p_{12}(\delta x_1 - \delta x_2) + \cdots + p_{1n}(\delta x_1 - \delta x_n)$. By similar definitions there are corresponding expressions for $\delta y_2, \cdots, \delta y_n$. The constants p_{ij} define a linear connected transformation; hence, by Lemma 1 this transformation has an inverse. But a linear transformation with an inverse has a unique inverse, so if $\delta y = 0$, then $\delta x = 0$. This completes the proof of Lemma 1.

Let us designate the inverse of transformation (2) by

$$
\begin{aligned}
x_1 &= R_1(y_1, \cdots, y_n),\\
&\cdots\cdots\cdots,\\
x_n &= R_n(y_1, \cdots, y_n).
\end{aligned}
$$

THEOREM 2. *The inverse of a connected transformation is a continuous transformation.*

PROOF. The proof of Lemma 1 with $\epsilon = 0$ shows that R is a bounded transformation. Thus, if $y' \to y$, then for some sub-sequence $x' = Ry'$ approaches a limit, say x. Now $y' = Px'$, so $y = Px$, since P is continuous. This uniquely defines x, so all sub-sequences have the same limit.

THEOREM 3. *In an n-dimensional connected transformation* $(n > 1)$ *delete the nth equation and let* $x_n = \text{constant}$ *in the rest. Then there remains an* $(n-1)$*-dimensional connected transformation.*

PROOF. The foundation for the new system is, if $x_n = c$,

$$
P_i'(t_1, \cdots, t_{n-1}) = P_i(t_1, \cdots, t_{n-1}, t_i - c), \qquad i = 1, \cdots, n-1,
$$

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so clearly conditions (a) and (b) are satisfied. The chain sequences which do not contain the integer n are left unchanged. A chain sequence of the form $i, a, \cdots, d, n, \cdots, h$ is replaced by the sequence $i, a, \cdots, d$. It follows that P'_{dd} is unbounded because P_{dn} is unbounded. Thus (c) is satisfied.

THEOREM 4. *If transformation* (2) *satisfies* (a) *and* (b) *but not* (c), *then it does not have a unique solution: For some integer k, x_k may be given an arbitrary value.*

PROOF. The integers $1, \cdots, n$ are divided into sets C and D. We put the integer i in C if and only if there exists a sequence of integers $a, b, \cdots, h$ such that each of the functions $P_{ia}, P_{ab}, \cdots, P_{gh}, P_{hh}$ is unbounded. It is to be noted that if this situation obtains each of $a, b, \cdots, h$ is also in C. Moreover, if i is in C and if P_j actually contains t_i, then j is in C because P_{ji} is not constant and the chain $P_{ji}, P_{ia}, \cdots, P_{hh}$ is made up of unbounded functions. An integer k is in D if and only if it is not in C; and in view of the preceding remarks, P_{kk} is a constant.

The variables x_i and the equations $y_i = P_i$ are also divided into two sets depending on whether i belongs to C or D. Considering the D variables as constants, the C equation forms a connected transformation in the C variables. To prove this, write down the foundation P' as in the proof of Theorem 3, and it is clear that conditions (a) and (b) are satisfied. The chain condition (c) is satisfied by construction. Hence, no matter what values are assigned to the variables D, the C equations can be satisfied. The C variables do not occur in the D equations. Moreover, the D variables occur only as paired differences in the D equations because P_{kk} is a constant if k is in D. Hence, adding the same constant to each of the D variables gives a family of solutions if there is one solution.

THEOREM 5. *If transformation* (2) *satisfies* (a) *and* (b) *but not* (c), *then it does not have a solution x for all y.*

PROOF. The theorem is clearly true for a one-dimensional transformation, so we proceed by induction. According to Theorem 4, we may set $x_k = 0$. Delete the kth equation. What is left is an $(n-1)$-dimensional transformation of the form (2) and (a) and (b) are satisfied. If (c) is also satisfied, x_i is uniquely determined; hence y_k is uniquely determined by the other y_i. If (c) is not satisfied, these equations are singular by the inductive hypothesis.

THEOREM 6. *In the inverse of an n-dimensional connected transforma-*

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tion $(n>1)$, *delete the nth equation and let* $y_n = \text{constant}$ *in the rest; then there remains the inverse of an* $(n-1)$*-dimensional connected transformation.*

PROOF. Thus $c = P_n(x_n - x_1, \cdots, x_n)$. Let $x_n - x_1 = z_1$, then $c = P_n(z_1, z_1 + (x_1 - x_2), \cdots, z_1 + x_1)$. By conditions (a), (b), and (c) it follows that P_n is a continuous increasing and unbounded function of z_1, so we may write $-z_1 = H_1(x_1, x_1 - x_2, \cdots, x_1 - x_{n-1})$. From conditions (a) and (b), it follows that $H_1(t_1, t_2, \cdots, t_{n-1})$ is a continuous function and that H_{1j} is either a constant or is an increasing function unbounded at $\pm\infty$. Substituting z_1 in the first equation gives $y_1 = P_1(x_1, x_1 - x_2, \cdots, H_1(x_1, x_1 - x_2, \cdots, x_1 - x_{n-1}))$. The first function of the new foundation thus is

$$P_1'(t_1, \cdots, t_{n-1}) = P_1(t_1, \cdots, t_{n-1}, H_1(t_1, \cdots, t_{n-1})).$$

Clearly P_1' satisfies (a) and (b). The same result follows, by symmetry, for $P_2', \cdots, P_{n-1}'$. But the transformation P' has a unique inverse; hence, by Theorem 4 or 5 it follows that (c) is satisfied.

THEOREM 7. *If* $x_1 = R_1(y_1, \cdots, y_n)$, $i = 1, \cdots, n$, *is the inverse of a connected transformation and* R_{ij} *indicates* R_i *as a function of* y_j *then* R_{ij} *and* $R_{ii} - R_{ji}$ *are either constant or increasing functions, unbounded at* $\pm\infty$; *moreover,* R_{ii} *is not constant.*

PROOF. By repeated application of Theorem 6, we note that $R_1(y_1, c_2, \cdots, c_n)$ is the inverse of a one-dimensional connected foundation for any choice of constants $c_2, \cdots, c_n$. Hence R_{11} is an increasing function, unbounded at $\pm\infty$. By symmetry the same is true of R_{ii}.

Likewise, $x_1 = R_1(y_1, y_2, c_3, \cdots, c_n)$ and $x_2 = R_2(y_1, y_2, c_3, \cdots, c_n)$ is the inverse of a connected transformation of the form $y_1 = P_1(x_1, x_1 - x_2)$ and $y_2 = P_2(x_2 - x_1, x_2)$. Suppose that y_1 takes and increase $\delta y_1 > 0$ and that $\delta y_2 = 0$. By the result just proved $\delta x_1 > 0$ and $\delta x_1 \to +\infty$ if $\delta y_1 \to +\infty$. Consider the second equation. If P_{21} is constant and P_{22} is not constant, $\delta x_2 = 0$ so $\delta(x_2 - x_1) = -\delta x_1 < 0$. If P_{22} is constant and P_{21} is not constant $\delta(x_2 - x_1) = 0$, so $\delta x_2 = \delta x_1 > 0$. If neither P_{21} nor P_{22} is constant either $\delta x_2 \leqq 0$ and $\delta(x_2 - x_1) \geqq 0$ or $\delta x_2 > 0$ and $\delta(x_2 - x_1) < 0$. The first possibility is incompatible with $\delta x_1 > 0$. In the latter case if $x_1 \to +\infty$ then x_2 can not remain bounded or y_2 would not be constant. By the same reasoning $(x_2 - x_1)$ is not bounded. A similar argument applies for $\delta x_1 < 0$, so this proves the theorem for R_{21} and $R_{11} - R_{21}$. By symmetry the theorem must hold for arbitrary indices.

2. **Linear connected transformations.** To put the connected trans-

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formation (3) into conventional matrix form, define a *connected matrix* $\|s_{ij}\|$ as

$$s_{ij} = -p_{ij}, \qquad i \neq j,$$

and

$$s_{ii} = p_{i1} + p_{i2} + \cdots + p_{in}.$$

Then equations (3) become $y_1 = \sum_{j=1}^{n} s_{ij}x_j$, $i = 1, \cdots, n$.

Sylvester [3][2] was the first to investigate linear transformations of this type, and he called the determinant $|s_{ij}|$ a *unisignant* determinant because of the nature of the following theorem.

THE SYLVESTER-BORCHARDT THEOREM. *The determinant $|s_{ij}|$ of an n-dimensional linear connected transformation is the sum of the products of the coefficients p_{ij} taken n at a time in such wise that the coefficients appearing in each product separately satisfy the chain condition.*

Later Maxwell [2] found that equations of the form (3) with $p_{ij} = p_{ji}$ define the steady flow of current in an electrical network of $n+1$ junctions, one of whose junction is held at zero potential. The other junctions have potentials $x_1, \cdots, x_n$, and the currents entering these junctions from outside are $y_1, \cdots, y_n$. The conductivity of the wire connecting the ith and jth junction is p_{ij}, $i \neq j$, while p_{kk} is the conductivity of the wire connecting the kth junction to the junction held at zero potential.

J. J. Thompson in an appendix to Chapter IV of the third edition of Maxwell's treatise stated a neat formula for the solution of the network equations. Equations of the form (3) with $p_{ij} = p_{ji}$ also appear in Maxwell's treatise in connection with the coefficients of capacity of neighboring conductors. Stieltjes [4] proved several theorems concerning these coefficients of capacity. Unfortunately these and later writers on electrical theory seem unaware of Sylvester's more profound treatment.

Let $\|r_{ij}\|$ be the inverse of the connected matrix $\|s_{ij}\|$. We shall now state some properties of these matrices which follow directly from the preceding theorems.

THEOREM 8. *The matrix $\|s'_{ij}\|$ corresponding to $p'_{ij} = p_{ij} + p_{ji}$ is positive definite.*

PROOF. If we let $p'_{ij} = p_{ij} + p_{ji}$, then $\|s'_{ij}\|$ is clearly a connected

[2] Numbers in brackets refer to the references cited at the end of the paper.

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matrix and so is nonsingular. The theorem is then obvious from the identity

$$2\sum_{1}^{n}\sum_{1}^{n} s'_{ij}x_ix_j = \sum_{1}^{n}\sum_{1}^{n} p'_{ij}(x_i - x_j)^2 + 2\sum_{1}^{n} p'_{ii}x_1^2.$$

THEOREM 9. *If* $n>1$ *then* $\|s_{nn}s_{ij}-s_{in}s_{nj}\|$, $i, j=1, \cdots, n-1$, *is an* $(n-1)$*-dimensional connected matrix.*

PROOF. According to Theorem 6, $y_i=\sum_1^n s_{ij}x_j$, $i=1, \cdots, n-1$, where x_n is defined by $0=\sum_1^n s_{nj}x_j$, is an $(n-1)$-dimensional connected transformation. Then $x_n=-\sum_1^{n-1} s_{nj}x_j/s_{nn}$ and $y_i=\sum_1^{n-1}(s_{nn}s_{ij} -s_{in}s_{nj})x_j/s_{nn}$. But since $s_{nn}>0$, the matrix may be multiplied by this constant without destroying properties (a), (b), and (c).

THEOREM 10. *If* $n>1$ *then* $\|r_{nn}r_{ij}-r_{in}r_{nj}\|$, $i, j=1, \cdots, n-1$, *is the inverse of an* $(n-1)$*-dimensional connected matrix.*

PROOF. The proof parallels that of Theorem 9 but using Theorem 3 instead of Theorem 6. That $r_{nn}>0$ is clear from Theorem 7.

THEOREM 11. *For* $i, j=1, \cdots, n$: (a) $r_{ij}\geqq 0$. (b) $r_{ii}>0$. (c) $r_{ii}\geqq r_{ji}$. (d) $r_{ij}r_{kk}\geqq r_{ik}r_{kj}$. (e) $r_{ii}r_{kk}>r_{ik}r_{ki}$, $i\neq k$.

PROOF. Theorem 7 gives (a), (b), and (c). Then (d) and (e) follow from Theorem 10.

In the case $n=2$, the inequalities of Theorem 11 are sufficient to define all inverse connected matrices. This suggests the problem of giving a direct definition of the inverse of a connected transformation; however, the writer has been unable to accomplish this.

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