STEINER'S FORMULAE ON A GENERAL S^{n+1}

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1. **Introduction.** Steiner's famous formulae on parallel curves and surfaces have attracted considerable interest recently, several mathematicians having developed various extensions of these theorems [3, 4, 6].[1] As stated by Steiner [5] these formulae have the following form:

THEOREM 1. *Let C be a convex curve in the plane of length L and area F, and let C_ρ be the curve parallel to C at a distance ρ from it (measured outward) with length L_ρ and area F_ρ; then*

$$L_\rho = L + 2\pi\rho, \qquad F_\rho = F + \rho L + \pi\rho^2.$$

THEOREM 2. *Let Σ be a convex surface in ordinary space of surface area S, enclosed volume V, and integrated mean curvature M; and let Σ_ρ be the surface parallel to Σ at a distance ρ from it (measured outward) with surface S_ρ and volume V_ρ; then:*

$$S_\rho = S + 2M\rho + 4\pi\rho^2, \qquad V_\rho = V + S\rho + M\rho^2 + 4\pi\rho^3/3.$$

We shall prove the following generalization of these results:

THEOREM 3. *Let S^{n+1} be a Riemann space of constant curvature, K, differentiable of class C^3 and complete in the sense of Hopf and Rinow. Let V^n be a hypersurface of S^{n+1} which is closed and bounding in S^{n+1} and of class C^3, and whose principal curvatures with respect to an outward normal are all negative. The area of V^n will be called A and its volume Vol. Its various mean curvatures (to be defined in §3) will be called M_i $(i=0, \cdots, n)$. Let V^n_ρ be a surface parallel to V^n at a distance measured along outward drawn geodesics where:*

$$\textit{for} \quad K > 0: 0 \leqq \rho \leqq \pi/2K^{1/2}; \quad \textit{and for} \quad K < 0: \rho \geqq 0.$$

Further let the area and volume of V^n_ρ be respectively A_ρ and Vol_ρ. *Then for $K>0$:*

$$A_\rho = \sum_{i=0}^{n} M_i(K^{-1/2} \sin [\rho K^{1/2}])^{n-i}(\cos [\rho K^{1/2}])^i,$$

$$\mathrm{Vol}_\rho = \mathrm{Vol} + \sum_{i=0}^{n} M_i \int_0^\rho (K^{-1/2} \sin [x^0K^{1/2}])^{n-i}(\cos [x^0K^{1/2}])^i dx^0;$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

And for $K<0$:

$$A_\rho = \sum_{i=0}^{n} M_i([-K]^{-1/2} \sinh [\rho(-K)^{1/2}])^{n-i}(\cosh [\rho(-K)^{1/2}])^i,$$

$$\mathrm{Vol}_\rho = \mathrm{Vol}$$

$$+ \sum_{i=0}^{n} M_i \int_0^\rho ([-K]^{-1/2} \sinh [x^0(-K)^{1/2}])^{n-i}(\cosh [x^0(-K)^{1/2}])^i dx^0.$$

Further simplifications of these formulae for special cases can be made by the use of the Gauss-Bonnet formula as developed by the author and A. Weil [1]. These results appear in §§3 and 5.

The methods used in deriving these results are similar to those of Vidal Abascal [6] who developed the special case of $n=1$ in a recent paper. Herglotz [4] has studied the same problem on spheres and on hyperbolic subspaces of pseudo-Euclidean space and has derived the above formulae for these restricted cases. Related formulae were developed by H. Weyl [7] in his study of the volume of tubes lying on spheres. Reference should also be made to the general study of parallel curves on a general two-dimensional surface of positive (non-constant) curvature by Fiala [2]. No such study is available for n-dimensional manifolds.

2. **Calculations.** In S^{n+1} we consider the geodesic parallel coordinate system in which the first fundamental form has the expression:

$$ds^2 = (dx^0)^2 + g_{\alpha\beta}dx^\alpha dx^\beta \qquad (\alpha, \beta = 1, \cdots, n). \tag{1}$$

This is so chosen that:

(a) $x^0=0$ is the hypersurface V^n;

(b) the curves $x^\alpha=$const. $(\alpha=1, \cdots, n)$ are geodesics normal to V^n, on which arc length is measured by x^0 positively outward from V^n;

(c) the positive orientation of S^{n+1} is given by the ordering $[x^0, x^1, \cdots, x^n]$ and that of V^n by $[x^1, \cdots, x^n]$.

Further let the values of $g_{\alpha\beta}$ for $x^0=0$ be denoted by $\gamma_{\alpha\beta}$, so that $\gamma_{\alpha\beta}$ are the components of the metric tensor V^n. When more machinery is available, we shall discuss the domain of S^{n+1} within which this is a proper coordinate system; but for the moment we assume that we are operating within such a domain.

If we assume that the normals to V^n in S^{n+1} are directed outward, we may recall the standard formula:

$$\left(\frac{\partial g_{\alpha\beta}}{\partial x^0}\right)_{x^0=0} = -\,\Omega_{\alpha\beta}/2, \tag{2}$$

where $\Omega_{\alpha\beta}$ are the coefficients of the second fundamental form of V^n relative to S^{n+1}. From (2) it follows that:

$$\left(\frac{\partial (g_{\alpha\beta})^{1/2}}{\partial x^0}\right)_{x^0=0} = -\,\frac{\Omega_{\alpha\beta}}{(\gamma_{\alpha\beta})^{1/2}}\,. \tag{3}$$

We now wish to calculate $g^{1/2}$ (where $g=\det |g_{\alpha\beta}|$) in terms of $\Omega_{\alpha\beta}$, $\gamma_{\alpha\beta}$, and x^0; for the integration of this quantity gives the desired formulae. To do this, we first consider a fixed point P on V^n and the geodesic $G(P)$ through P normal to V^n (its arc length is x^0). By a linear transformation (constant coefficients) of x^α $(\alpha=1, \cdots, n)$ we can find a new coordinate system on V^n, $\bar{x}^\alpha = L^\alpha(x)$, in which the components of the first and second fundamental forms of V^n at P reduce to sums of squares, namely $\delta_{\alpha\beta}$ and $\bar{\Omega}_{\alpha\beta}$ (where $\bar{\Omega}_{\alpha\beta}=0$, $\alpha\neq\beta$).

Now in S^{n+1} apply the coordinate transformation:

$$\bar{x}^0 = x^0,$$
$$\bar{x}^\alpha = L^\alpha(x^\beta), \qquad \alpha, \beta = 1, \cdots, n.$$

The new line element is:

$$ds^2 = (dx^0)^2 + \bar{g}_{\alpha\beta}\,d\bar{x}^\alpha d\bar{x}^\beta \tag{4}$$

where $\bar{g}_{\alpha\beta}=\delta_{\alpha\beta}$ at P, and $\partial\bar{g}_{\alpha\beta}/\partial x^0=0$, $\alpha\neq\beta$ at P. We now wish to prove the lemma:

LEMMA. *In the $\bar{x}$ coordinate system, $\bar{g}_{\alpha\beta}=0$ $(\alpha\neq\beta)$ for all points on $G(P)$.*

From (4) it follows that anywhere in S^{n+1}

$$\bar{R}_{\alpha 00\beta} = \frac{1}{2}\left(\frac{\partial^2 \bar{g}_{\alpha\beta}}{\partial x^0 \partial x^0}\right) - \frac{1}{4}\left(\bar{g}^{\gamma\delta}\,\frac{\partial \bar{g}_{\gamma\alpha}}{\partial x^0}\,\frac{\partial \bar{g}_{\beta\delta}}{\partial x^0}\right). \tag{5}$$

But since S^{n+1} is of constant curvature:

$$\bar{R}_{\alpha 00\beta} = -\,K\bar{g}_{\alpha\beta}; \tag{6}$$

so

$$\frac{1}{2}\left(\frac{\partial^2 \bar{g}_{\alpha\beta}}{\partial x^0 \partial x^0}\right) - \frac{1}{4}\,\bar{g}^{\gamma\delta}\left(\frac{\partial \bar{g}_{\gamma\alpha}}{\partial x^0}\,\frac{\partial \bar{g}_{\beta\delta}}{\partial x^0}\right) + K\bar{g}_{\alpha\beta} = 0. \tag{7}$$

For $\alpha\neq\beta$, equations (7) may be considered to be differential equations in $\bar{g}_{\alpha\beta}$ $(\alpha\neq\beta)$ whose coefficients involve constants and $\bar{g}_{uu}$ and

$\partial\bar{g}_{uu}/\partial x^0$. Taking these last functions as known, we consider the unique solution of (7) along $G(P)$ with initial values at P as given above, namely:

$$(\bar{g}_{\alpha\beta})_P = 0; \qquad \left(\frac{\partial\bar{g}_{\alpha\beta}}{\partial x^0}\right)_P = 0 \qquad \text{for } \alpha \neq \beta. \tag{8}$$

Now $\bar{g}_{\alpha\beta}=0$ $(\alpha\neq\beta)$ is a solution of (7) satisfying the initial conditions; hence it is the only solution, and the lemma is proved.

Returning now to (7) with $\alpha=\beta$, we find that as a result of the lemma the following equations hold along $G(P)$:

$$\frac{\partial^2(\bar{g}_{\alpha\alpha})^{1/2}}{\partial x^0\partial x^0} + K(\bar{g}_{\alpha\alpha})^{1/2} = 0. \tag{9}$$

Integrating (9) and taking (3) into account we find that:

$$(\bar{g}_{\alpha\alpha})^{1/2} = -\bar{\Omega}_{\alpha\alpha}(K^{-1/2}\sin[x^0K^{1/2}]) + \cos[x^0K^{1/2}]. \tag{10}$$

Hence along $G(P)$:

$$\bar{g}^{1/2} = \prod_{\alpha=1}^{n}\left\{-\bar{\Omega}_{\alpha\alpha}(K^{-1/2}\sin[x^0K^{1/2}]) + \cos[x^0K^{1/2}]\right\}. \tag{11}$$

Returning to the original coordinate system we find that (11) transforms into:

$$g^{1/2} = \gamma^{-1/2}\det\left| -\Omega_{\alpha\beta}(K^{-1/2}\sin[x^0K^{1/2}]) + \gamma_{\alpha\beta}\cos[x^0K^{1/2}]\right| \tag{12}$$

where $\gamma=\det|\gamma_{\alpha\beta}|$. Now equation (12) is valid on every geodesic $G(P)$ and hence holds throughout the entire region under consideration.

It is now possible to discuss the domain of validity of formula (12). First we must require that $g^{1/2}>0$; this limits x^0 to be less than the minimum distance to the first conjugate point on any geodesic $G(P)$. An effective way to do this (but not the most general) is to make the assumptions stated in Theorem 3. Even with this restriction it is still possible that two distinct geodesics $G(P_1)$ and $G(P_2)$ will intersect on S^{n+1}, and hence cause a singularity in the coordinate system. This awkward complication may be avoided by supposing that we are dealing not with S^{n+1} but with a covering surface of it which puts these geodesics on separate sheets. We make this assumption, and thus in dealing with the volumes discussed in this paper we agree to count overlapping volumes with the necessary (finite) multiplicity. This means that our "parallel surfaces" are not necessarily true parallels in the sense of being the locus of points at a constant minimum distance from V^n, but they are more properly called "geodesic parallel surfaces."

3. **Results.** We are now in a position to prove Theorem 3. For

$$A_\rho = \int_{V^n} (g^{1/2})_{x^0=\rho} dx^1 \cdots dx^n \tag{13}$$

and

$$\mathrm{Vol}_\rho = \mathrm{Vol} + \int_{V^n} \left\{ \int_0^\rho g^{1/2} dx^0 \right\} dx^1 \cdots dx^n \tag{14}$$

where $g^{1/2}$ is given by (12).

To simplify these results consider the expression:

$$\left| -\Omega_{\alpha\beta} + \lambda\gamma_{\alpha\beta} \right| \equiv \theta_0 + \theta_1\lambda + \theta_2\lambda^2 + \cdots + \theta_n\lambda^n \tag{15}$$

where the θ_i are thus defined. Further let

$$M_i = \int_{V^n} \theta_i \gamma^{-1/2} dx^1 \cdots dx^n. \tag{16}$$

We call M_i the ith mean curvature of V^n in S^{n+1}. Since $\theta_n = \gamma$, $M_n = A$. For $K > 0$ the expansion of (13) and (14) using the notation of (16) gives the formulae of Theorem 3. These formulae are indeed valid for $K < 0$, but in this case they appear to involve complex numbers. By introducing the hyperbolic functions we find that in fact the entire expression is real, and so explicit formulae for this case are given in Theorem 3.

Further simplification can be obtained in certain cases by the use of the Gauss-Bonnet formula. To prepare for this we introduce the following notation:

$$\tag{17} \begin{aligned} \phi_i(\rho) &= \int_0^\rho (K^{-1/2} \sin [x^0 K^{1/2}])^{n-i} (\cos [x^0 K^{1/2}])^i dx^0; \qquad K > 0, \\ \phi_i(\rho) &= \int_0^\rho ([-K]^{-1/2} \sinh [x^0(-K)^{1/2}])^{n-i} (\cos [x^0(-K)^{1/2}])^i dx^0; \\ &\qquad K < 0, \end{aligned}$$

$$C_i = K^{(n+1)/2} \phi_i \left(\frac{\pi}{2K^{1/2}} \right) = \frac{\omega^{n+1}}{\omega^i \omega^{n-i}} K^{i/2} \tag{18}$$

where ω^j is the surface area of a j-dimensional unit sphere (its surface is of j dimensions) and $\omega^0 = 2$. In this notation the Gauss-Bonnet formula can be written:

$$\text{(19)}\quad \text{For } n \text{ even: } M_0C_0 + M_2C_2 + \cdots + M_nC_n = -\omega^{n+1}\chi'/2 = \omega^{n+1}\chi/4;$$

$$\text{(20)}\quad \text{For } n \text{ odd: } M_0C_0 + M_2C_2 + \cdots + M_{n-1}C_{n-1} + K^{(n+1)/2}\,\mathrm{Vol} = \omega^{n+1}\chi'/2;$$

where χ' is the inner characteristic of the volume bounded by V^n and χ is the characteristic of V^n itself. These relations may be used to eliminate M_0 from the formulae of Theorem 3. We give two examples.

Example 1. When $n=1$, (20) gives: $M_0=2\pi-FK$, where we have written F instead of Vol to represent the area inclosed by the given curve. This leads to the results of Vidal Abascal which generalize the formulae of Theorem 1. Using the notations of Theorem 1, the calculations are as follows:

$$\begin{aligned} F_\rho &= F + \phi_0 M_0 + \phi_1 M_1 \\ \text{(21)}\qquad &= F + K^{-1}[1 - \cos(\rho K^{1/2})][2\pi - FK] + K^{-1/2}\sin[\rho K^{1/2}]L \\ &= F\cos[\rho K^{1/2}] + K^{-1/2}\sin[\rho K^{1/2}]L + 2\pi K^{-1}(1 - \cos[\rho K^{1/2}]). \end{aligned}$$

The derivative of this expression with respect to ρ gives the corresponding formula for L_ρ.

Example 2. When $n=2$, (19) gives:

$$\text{(22)}\qquad M_0 + AK = -4\pi x' = 2\pi\chi.$$

This leads to the formulae:

$$\text{(23)}\qquad \begin{aligned} A_\rho = A &+ M_1K^{-1/2}\sin(\rho K^{1/2})\cos(\rho K^{1/2}) \\ &+ 2\pi\chi K^{-1}\sin^2(\rho K^{1/2}), \end{aligned}$$

$$\text{(24)}\qquad \begin{aligned} \mathrm{Vol}_\rho = \mathrm{Vol} &+ A_\rho + M_1(2K)^{-1}\sin^2(\rho K^{1/2}) \\ &+ \pi\chi K^{-1}(\rho - K^{-1/2}\sin(\rho K^{1/2})\cos(\rho K^{1/2})). \end{aligned}$$

These are generalizations of those given in Theorem 2.

4. **Formulae for** $K=0$. In the preceding discussion it has been explicitly assumed that $K\neq 0$. To derive similar formulae for $K=0$ we can repeat the above discussion under this assumption; and we also get the same result by taking the limit of the formulae of Theorem 3 as $K\to 0$. The results are:

$$\text{(25)}\qquad A_\rho = \sum_{i=0}^{n} M_i\rho^{n-i},$$

$$\text{(26)}\qquad \mathrm{Vol}_\rho = \mathrm{Vol} + \sum_{i=0}^{n}(n-i+1)^{-1}M_i\rho^{n-i+1}.$$

In this case the Gauss-Bonnet theorem says that:

$$(27)\qquad M_0 = \begin{cases} -\omega^n\chi' = \omega^n\chi/2, & n \text{ even,} \\ +\omega^n\chi', & n \text{ odd.} \end{cases}$$

Combining this with (25) and (26) and rearranging we have:

$$(28)\qquad A_\rho = A + \sum_{i=1}^{n-1} M_{n-i}\rho^i + \rho^n \begin{cases} \omega^n\chi/2, & n \text{ even,} \\ \omega^n\chi', & n \text{ odd,} \end{cases}$$

and

$$(29)\qquad \mathrm{Vol}_\rho = \mathrm{Vol} + A\rho + \sum_{i=2}^{n} \frac{M_{n-i+1}\rho^i}{i} + \frac{\rho^{n+1}}{n+1} \begin{cases} \omega^n\chi/2, & n \text{ even,} \\ \omega^n\chi', & n \text{ odd.} \end{cases}$$

In comparing these results with Theorem 2, we note that the M of Theorem 2 equals $M_1/2$ as here defined.

5. **Polar surfaces.** When $K>0$, the formulae (18), (19) and (20) suggest that we consider the "polar" surface to V^n; that is, the surface at a distance $\pi/(2K^{1/2})$ from V^n. Since this surface may lie outside the domain of validity of our coordinate system, the application of Theorem 3 will not in general give correct results. Formal application of the formulae of Theorem 3, however, does give answers which may be interpreted as the algebraic area and volume of the polar surface respectively. This means that in carrying out the integrations we have algebraically combined the positive and negative portions of the result and thus obtained their algebraic sum. Using a subscript, P, to indicate the polar surface, we find that:

$$(30)\qquad A_P = M_0K^{-n/2}.$$

Thus for $n=1$:

$$(31)\qquad L_P = (2\pi - FK)/K^{-1/2}.$$

And for $n=2$:

$$(32)\qquad A_P = (2\pi\chi - AK)/K \quad \text{or} \quad A + A_P = 2\pi\chi/K.$$

This is a generalization of a result due to Blascke [4] in which he assumes that V^2 is a topologic sphere.

The Gauss-Bonnet formula suggests that we consider the "volume" (here we mean algebraic volume) bounded by the two polar surfaces at distances $+\pi/(2K^{1/2})$ and $-\pi/(2K^{1/2})$ respectively. Designating this volume by the symbol $\mathrm{Vol}\big|_{-P}^{P}$ we find that for n even:

$$(33)\qquad \mathrm{Vol}\big|_{-P}^{P} = 2(M_0C_0 + M_2C_2 + \cdots + M_nC_n)/K^{(n+1)/2}.$$

Or from the Gauss-Bonnet theorem it follows that:

$$\mathrm{Vol}\Big|_{-P}^{P} = \frac{1}{2}\,\frac{\omega^{n+1}}{K^{(n+1)/2}}\,\chi. \tag{34}$$

If V^n is a topologic sphere (n even) we see that the volume $\mathrm{Vol}\big|_{-P}^{P}$ equals the surface area of an $(n+1)$-dimensional sphere of radius $K^{-1/2}$.

The analogous result for n odd is obtained by considering the "doubly polar surface" to V^n, namely the one at a distance of $\pi/K^{1/2}$ from V^n. Designating the volume between V^n and this surface by $\mathrm{Vol}\big|_0^{2P}$ we find that for n odd:

$$\mathrm{Vol}\Big|_0^{2P} = 2(M_0C_0 + M_2C_2 + \cdots + M_{n-1}C_{n-1})/K^{(n+1)/2}. \tag{35}$$

Or from the Gauss-Bonnet theorem it follows that:

$$\mathrm{Vol}\Big|_0^{2P} = \frac{\omega^{n+1}}{K^{(n+1)/2}}\,\chi' - 2\,\mathrm{Vol} \tag{36}$$

or

$$2\,\mathrm{Vol} + \mathrm{Vol}\Big|_0^{2P} = \frac{\omega^{n+1}}{K^{(n+1)/2}}\,\chi'. \tag{37}$$

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