Now, by the original choice of U, V_0 has the property required by the theorem.

We remark that the same proof holds, with trivial modifications, for homology local connectivity.

It is an open question whether the neighborhood V can be required to be connected.

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UNCONDITIONAL CONVERGENCE IN BANACH SPACES

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Introduction. This note investigates an apparent generalization of unconditionally convergent series $\sum x_i$ in weakly complete Banach spaces. A series of elements with x_i in E is said to be unconditionally convergent if for every variation of sign $\epsilon_l = \pm 1$, $\sum_1^m \epsilon_l x_l$ is convergent. This formulation of the definition of unconditional convergence is equivalent to that given by Orlicz [4].[1] We call $\sum x_i$ unconditionally summable if there exists a finite row Toeplitz matrix (b_{ik}) such that for every variation of sign $\sigma_i = \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l x_l$ converges. The fact that unconditional summability implies unconditional convergence is established in this note. Finally, applications to orthogonal functions are presented.

Preliminary lemmas. In what follows, b_{ik} will denote an arbitrary finite row Toeplitz matrix.

LEMMA 1. If $S_n(\theta) = \sum_1^m a_l r_l(\theta)$ *converges to an essentially bounded function* $f(t)$, *then* $\left| \sum_1^m a_n r_n(\theta) \right| \leqq c$ *almost everywhere.* ($r_n(\theta)$ *denote the Rademacher functions.*)

PROOF. This is an immediate consequence of the result that

$$\left(\int_0^1 \left(\max_{1 \leqq n \leqq m} \left| \sum_1^n a_l r_l(\theta) \right| \right)^p d\theta \right)^{1/p} \leqq C \left(\int_0^1 \left| S_m(\theta) \right|^p d\theta \right)^{1/p}, \qquad 1 \leqq p \leqq \infty, \tag{1}$$

Received by the editors April 30, 1947.

[1] Numbers in brackets refer to the references cited at the end of the paper.

for all m [3]. For, applying (1) for $p=\infty$, we get

$$\left| \sum_1^m a_l r_l(\theta) \right| \leqq \max_n \left| \sum_1^n a_l r_l(\theta) \right| \leqq C \left| \sum_1^\infty a_l r_l(\theta) \right| \leqq cf(\theta) \leqq \gamma.$$

LEMMA 2. *If* $\sigma_i = \sum_{k=1}^{m_i} b_{ik} \sum_{l=1}^{k} \epsilon_l a_l$ *converges for every* $\epsilon_l = \pm 1$, *then* $\sum_1^\infty |a_l| < \infty$.

PROOF. We first establish that if $A_i = \sum_{k=1}^{m_i} A_{ik}\epsilon_k$ converges for every variation of sign $\epsilon_k = \pm 1$, then $\sum_{k=1}^{n_i} |A_{ik}| \leqq C$ for every i. If we observe that $\{(A_{ik}) = x_i\}_i$ is a sequence of elements in (l) (space of absolutely convergent series), then the hypothesis implies that x_i converges weakly for all functionals f of the form $f = \{\epsilon_k\}$. A result of Banach [1] implies x_i are strongly convergent and hence $\|x_i\| = \sum_{k=1}^{n_i} |A_{ik}| \leqq C$. We now complete the proof. Since

$$\sigma_i = \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l a_l = \sum_{l=1}^{n_i} \epsilon_l \left[a_l \sum_{k=l}^{n_i} b_{ik} \right] = \sum_{l=1}^{n_i} \epsilon_l A_{il}$$

converges, we have, in view of the preceding, that

$$\left| \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l a_l \right| \leqq \sum_{l=1}^{n_i} \left| a_l \sum_{k=l}^{n_i} b_{ik} \right| = \sum_{l=1}^{n_i} |A_{il}| \leqq C. \tag{2}$$

If we replace the ϵ_l by $r_l(\theta)$, then the hypothesis states that $\sigma_i(\theta)$ converges for every θ. In virtue of a known result [6], this implies that $S_m(\theta) = \sum_1^m a_l r_l(\theta)$ converges almost everywhere. This fact combined with (2) yields easily by Lemma 1 that almost everywhere

$$\left| \sum_1^m a_n r_n(\theta) \right| \leqq C, \qquad n = 1, 2, \cdots.$$

Using the independence of the Rademacher functions, we obtain for $|\lambda_k| \leqq 1$ that

$$\sum_{n=1}^m a_n \lambda_n = \int_0^1 \left(\sum_1^m a_n r_n(\theta) \right) \prod_1^m (1 + \lambda_k r_k(\theta)) d\theta$$
$$\leqq C \int_0^1 \prod_1^n (1 + \lambda_k r_k(\theta)) d\theta \leqq C.$$

Choosing $\lambda_n = \text{sign } a_n$, we get $\sum |a_n| < \infty$, which completes the proof.

The hypothesis requiring every set $\{\epsilon_l\}$ in the lemma is necessary. For there exist numbers a_l with $\sum a_l^2 < \infty$, $\sum |a_l| = \infty$, and $\sum a_l r_l(\theta)$ convergent almost everywhere, for example $a_l = 1/l$.

Another result needed in the same direction is the following:

LEMMA 3. *If for almost all variation of sign*

$$\left| \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l a_l \right| \leqq C,$$

then $\sum |a_l| < \infty$.

PROOF. In view of the hypothesis and the proof of the preceding lemma, it will be sufficient to show that $\sigma_i = \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l a_l$ converges almost everywhere. We proceed to show this. The hypothesis implies that for $\sigma_i(\theta) = \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} r_l(\theta) a_l$, we have for any $p > 1$

$$\int_0^1 |\sigma_i(\theta)|^p d\theta \leqq \gamma.$$

From a result of Banach and Saks [5], we infer

$$\lim_{n,m\to\infty} \int_0^1 \left| \frac{1}{m} \sum_{k=1}^{m} \sigma_{i_k}(\theta) - \frac{1}{n} \sum_{k=1}^{n} \sigma_{i_k}(\theta) \right|^p d\theta = 0. \tag{3}$$

This can be represented as a new Toeplitz matrix E operating on $\sum_1^m a_l r_l(\theta)$ with $E(S_m(\theta)) = \{\sigma_n'\} = \{(1/n) \sum_{k=1}^{n} \sigma_{i_k}\}$. Consequently, (3) states that

$$\int_0^1 |\sigma_n'(\theta) - \sigma_m'(\theta)|^p d\theta \to 0. \tag{4}$$

But this yields the existence of a subsequence $\sigma_{n_k}'(\theta)$ which converges almost everywhere. As above, this can be represented as $\{\sigma_{n_k}'(\theta)\} = E'\{\sigma_n'(\theta)\} = E'E\{S_n(\theta)\}$ where $E'E$ is a new Toeplitz matrix. Hence, we have exhibited a Toeplitz matrix which sums $S_m(\theta)$. As in Lemma 1 this implies that $S_m(\theta)$ converges almost everywhere. Q.E.D.

The theorems. We now establish several results on unconditional convergent series in a Banach space E.

We assume that E is weakly complete.

THEOREM 1. *If for* $x_i \in E$, $\sum x_i$ *is summable weakly by a Toeplitz matrix for every variation of sign, that is,*

$$\sigma_i = \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l x_l$$

converges weakly as elements, then $\sum x_l$ *is unconditionally convergent.*

PROOF. In view of the fact that E is weakly complete, it is sufficient to demonstrate that $\sum|f(x_l)| < \infty$ for every functional f [see [4]). If f denotes an arbitrary functional and $a_i = f(x_i)$, the hypothesis implies that, for every variation of sign $\epsilon_l = \pm 1$, $\sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l a_l$ converges. If we apply Lemma 2, we obtain that

$$\sum |a_l| < \infty \quad \text{or} \quad \sum |f(x_l)| < \infty.$$

The following similar result can be established with the use of Lemma 3.

THEOREM 2. *If E is weakly complete and $\| \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l x_l \| \leqq C$ for almost all variation of signs, then $\sum x_l$ is unconditionally convergent.*

A final result in this connection is the following. We assume that E is weakly complete.

THEOREM 3. *If for every rearrangement there exists a positive Toeplitz matrix with $\sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} x_{q(l)}$ convergent for x_l in E, then $\sum x_l$ converges unconditionally.*

PROOF. If $f(x_l) = a_l$, we assert that $\sum |a_l| < \infty$. Otherwise there exists a rearrangement with $\sum a_{n(q)} = \infty$ and $\sum_1^m a_{n(q)} > 0$ for every m. This implies that, for any positive Toeplitz matrix, $\sum_{k=1}^{n_i} b_{ik} \sum_1^k a_{n(q)} \to \infty$ which contradicts the hypothesis. The remainder of the proof is straightforward.

Applications. We shall now apply these results to complete orthogonal systems of functions. It is customary in treating bounded orthogonal systems to assume that the Lebesque kernel is summable by a finite row Toeplitz matrix, that is

$$\int_0^1 \left| \sum_{k=1}^{ni} b_{ik} \sum_{l=1}^{k} \phi_l(t)\phi_l(\theta) \right| dt \leqq C$$

[2] almost everywhere in θ. We make this assumption and establish the following theorem.

THEOREM 4. *If for every variation of sign $\epsilon_l = \pm 1$, $\{\epsilon_l a_l\}$ are the Fourier coefficients with respect to $\{\phi_k(t)\}$ of a function in $L^p (p \geqq 1)$, then $\{d_l a_l\}$ are the Fourier coefficients of a function in L^p for every sequence of numbers d_k with $|d_k| \leqq 1$.*

We first demonstrate a lemma.

LEMMA. *If $\sum x_i$ converges unconditionally, then $\| \sum_1^m d_l x_l \| \leqq C$ for all d_l with $|d_l| \leqq 1$.*

PROOF. The hypothesis implies that

$$\left\| \sum_1^m \epsilon_l x_l \right\| \leqq C \tag{5}$$

for all m and $\epsilon_l = \pm 1$. For each m and $l = 1, \cdots, m$ with $|d_l| \leqq 1$ there exists a functional f_m with $|f_m| = 1$ such that

$$\left\| \sum_{l=1}^m d_l x_l \right\| = f_m\left(\sum_1^m d_l x_l \right). \tag{6}$$

We obtain now using (5) and (6)

$$\begin{aligned} \left\| \sum_1^m d_l x_l \right\| &= \left| \sum_1^m d_l f_m(x_l) \right| \leqq \sum_1^m | f_m(x_l) | \\ &= \sum_1^m \epsilon_l f_m(x_l) = f_m\left(\sum_1^m \epsilon_l x_l \right) \\ &\leqq | f_m | \left\| \sum_1^m \epsilon_l x_l \right\| \leqq C. \end{aligned}$$

PROOF OF THEOREM 4. Let $\sigma_i(t) = \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l a_l \phi_l(t)$, then in virtue of the criterion that a sequence of real numbers be the Fourier coefficients of a function in L^p [2], we conclude that $\sum a_l \phi_l(t)$ is unconditionally summable in L^p. Applying Theorem 1, we obtain that $\sum a_l \phi_l(t)$ is unconditionally convergent. The above lemma implies that $\sum_1^m d_l a_l \phi_l(t)$ is uniformly bounded in L^p for any fixed set of d_l. Applying the criterion for determining Fourier coefficient in L^p gives us the result.

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