NOTE ON THE LOCATION OF THE CRITICAL POINTS OF HARMONIC FUNCTIONS

J. L. WALSH

By a limiting process, a theorem recently proved by the writer can be generalized, and yields a new result with interesting applications which we wish to record here. We take as point of departure[1] the following theorem.

THEOREM 1. *Let the region R of the extended (x, y)-plane be bounded by a finite number of mutually disjoint Jordan curves $C_0, C_1, C_2, \cdots, C_n$. Let the function $u(x, y)$ be harmonic in R, continuous in the corresponding closed region, equal to zero on C_0 and to unity on $C_1, C_2, \cdots, C_n$. Denote by R_0 the region bounded by C_0 containing the curves $C_1, C_2, \cdots, C_n$ in its interior; define noneuclidean straight lines in R_0 as the images of arcs of circles orthogonal to the unit circle, when R_0 is mapped conformally onto the interior of the unit circle.*

If Π is any non-euclidean convex region in R_0 which contains all the curves $C_1, C_2, \cdots, C_n$, then Π also contains all critical points of $u(x, y)$ in R.

We extend Theorem 1 by admitting arcs of C_0 on which $u(x, y)$ is prescribed to take the value unity, and also by admitting the intersection of curves $C_1, C_2, \cdots, C_n$ with C_0:

THEOREM 2. *Let the region R be bounded by the whole or part of the Jordan curve C_0, and by mutually disjoint Jordan arcs or curves $C_1, C_2, \cdots, C_n$ in the closed interior of C_0; some or all of the latter arcs or curves may have points in common with C_0. Let a finite number of arcs $\alpha_1, \alpha_2, \cdots, \alpha_m$ of C_0 belong to the boundary of R and be mutually disjoint. Let the function $u(x, y)$ be harmonic and bounded in R, and take continuously the boundary values unity on $C_1, C_2, \cdots, C_n, \alpha_1, \alpha_2, \cdots, \alpha_m$ and zero in the remaining boundary points of R, except that in points common to C_0 and $C_1+C_2+ \cdots +C_n$ and in end points of the α_j, no continuous boundary value is required. Denote by R_0 the region bounded by C_0 containing R, and define non-euclidean straight lines in R_0 by mapping R_0 onto the interior of a circle. If Π is any closed region in the closure of R_0 which is non-euclidean convex and which contains $C_1+C_2+ \cdots +C_n+\alpha_1+\alpha_2+ \cdots +\alpha_m$, then Π contains all critical point of $u(x, y)$ in R.*

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[1] Bull. Amer. Math. Soc. vol. 52 (1946) pp. 346–347.

Theorem 2 may be proved by mapping R_0 onto the interior of the unit circle; we retain the original notation. The region R can be approximated by a region R' bounded by C_0 and by Jordan curves $C_1', C_2', \cdots, C_n', \alpha_1', \alpha_2', \cdots, \alpha_m'$ in R_0 which are mutually disjoint and disjoint with C_0 and which respectively approximate $C_1, C_2, \cdots, C_n, \alpha_1, \alpha_2, \cdots, \alpha_m$. Let the function $u'(x, y)$ be harmonic in R', continuous in the corresponding closed region, zero on C_0 and unity elsewhere on the boundary of R'. Then as R' suitably approaches R, the variable function $u'(x, y)$ approaches $u(x, y)$ throughout R, uniformly on any closed set interior to R; we omit the proof. Any critical point of $u(x, y)$ interior to R is a limit point of critical points of the variable function $u'(x, y)$, so Theorem 2 follows from Theorem 1.

A further general result has recently been established[2] for the case $n=0$, which constructs Π in R_0 not by joining the ends of each arc of C_0 in the complement of the set α_j by a non-euclidean line but by similarly joining the ends of each double arc composed of an α_j plus one of the adjoining arcs of C_0 complementary to the set $\alpha_1+\alpha_2+ \cdots +\alpha_m$. It is still true (we shall refer to this result as Theorem 3) that Π contains all critical points in R of the corresponding harmonic function $u(x, y)$.

Theorem 3 is more powerful than Theorem 2 for the case $n=0$, but requires for its application essentially the use of a specific conformal map, and the latter quality may be an advantage or a disadvantage. It is an indication of the power of Theorem 2 that in the application of it to a given configuration, with or without the auxiliary use of conformal mapping, there may obviously be some arbitrariness in the notation, especially as to what shall be chosen as the region R_0. So far as convenience is concerned, it is desirable to choose simple configurations, where the totality or useful subset of non-euclidean lines is easily determined. It is also well to choose R_0 in such a way that the point set $C_1+ \cdots +C_n+\alpha_1+ \cdots +\alpha_m$ is as nearly non-euclidean convex as possible. But *if the aim is precision, the larger* R_0 *the better*, as we proceed to indicate in a special but typical case.

In Theorem 2, let C_0 be the unit circle in the z-plane, $n=2$, $m=0$, with C_1 and C_2 mutually disjoint Jordan arcs whose end points lie on C_0 and whose interior points lie interior to C_0. Let the subregion R of the interior of C_0 be bounded by C_1, C_2, and two appropriate arcs of C_0. In the actual application of Theorem 2, we can choose R_0 as the interior of C_0, or as the region R_1 containing R bounded by C_1

[2] J. L. Walsh, Proc. Nat. Acad. Sci. U.S.A. vol. 33 (1947) pp. 18–20.

and a suitable arc of C_0, or as the region R_2 containing R bounded by C_2 and a suitable arc of C_0, or as R. We now show as a general indication but without a complete rigorous proof that among these choices *the most precise results are obtained by choosing R_0 as the interior of C_0.*

Map (for instance) the region R onto the interior of the unit circle in the w-plane. Let α_z be an arbitrary arc of C_0 belonging to the boundary of R, which corresponds to the arc α_w in the w-plane. Let α_z' be the circular arc having the same end points as α_z, orthogonal to α_z, and whose interior points lie interior to C_0; we assume α_z' to lie in the closure of R. Let α_w' be the circular arc having the same end points as α_w, orthogonal to α_w, and whose interior points lie in $|w|<1$. The arcs α_z' and α_w' determine the respective non-euclidean geometries in the z-plane and w-plane, and it follows from a general theorem due to R. Nevanlinna[3] that *the region bounded by α_z and α_z' contains every point of the image of the region bounded by α_w and α_w'*. Corresponding to every arc α_z belonging to the boundary of R and on which the prescribed boundary value of $u(x, y)$ is zero, and to the adjacent arc α_z', with no point of C_1 or C_2 in the lens-shaped region between α_z and α_z' there exists in the w-plane an arc α_w whose end points correspond to those of α_z under the conformal map, such that the interior points of the arc α_w' lie interior to the lens-shaped region bounded by α_w and the image of α_z'. It follows that if we neglect arcs α_z' that cut C_1 or C_2 in R, then in this particular case Theorem 2 can be more favorably applied by choosing R_0 as the interior of C_0, that is to say, as large as possible.

The remark just made is of fairly general application. Moreover, in the specific case used, the interior of the given C_0 may be enlarged, without altering R or $u(x, y)$, by adding to R_0 regions adjacent to the arcs A of the given C_0 bounded by the end points of C_1 and C_2, the arcs A not being part of the boundary of R. Indeed, we may even adjoin an infinitely many sheeted logarithmic Riemann surface along each arc A; this is equivalent to mapping onto the interior of the unit circle the region R plus auxiliary regions, so that with the omission of two exceptional points the circumference of the unit circle corresponds only to that part of the boundary of R on which the prescribed boundary value of $u(x, y)$ is zero. The image of C_1 (and likewise of C_2) under this map is a Jordan curve which except for a single point lies interior to the unit circle.

Still another instructive kind of conformal map can be used, namely to map $R+C_1+C_2$ onto the interior of the unit circle in such a way

[3] Eindeutige analytische Funktionen, Berlin, 1936, p. 51.

that C_1 and C_2 correspond to radial slits, while the part of the boundary of R on which $u(x, y)$ has the prescribed boundary value zero corresponds to the whole circumference less two points. Here the region Π of Theorem 2 may degenerate to a line segment.

Another indication of the power of Theorem 2 is the following. Let C_0 be the unit circle, $n=1$ with C_1 a concentric circle of radius $r_1<1$; let an arc α (not the whole circle) of C_0 contain all the arcs α_j. By a conformal map of the universal covering surface of R onto the unit circle and application of Theorem 3 extended to the case of an infinite number of arcs, α_j, it follows (loc. cit. footnote 2) that in the original plane no critical points of $u(x, y)$ lie in the annulus $r_1<r<r_1^{1/2}$; a second annulus $r_2<r<1$ free from critical points can also be determined by this method. By Theorem 2, any circle cutting C_0 orthogonally in two points of the complement of α and containing in its interior no point of α or of C_1 contains in its interior no critical point of $u(x, y)$. In all, these conclusions may leave only a very small subregion of R as the portion in which the critical points of $u(x, y)$ lie.

We continue with a generalization of this result, a further application of Theorem 2:

THEOREM 4. *Let R be a region bounded by the whole of the Jordan curve C_0, by the whole or part of the Jordan curve C_1 disjoint from C_0, and by mutually disjoint Jordan arcs or curves $C_2, C_3, \cdots, C_n$ in the closed interior of the annulus R_0 bounded by C_0 and C_1; some or all of the latter arcs and curves are permitted to have points in common with C_1, but none with C_0. Let a finite number of arcs $\beta_1, \beta_2, \cdots, \beta_m$ of C_1 be part of the boundary of R and mutually disjoint. Let the function $u(x, y)$ be harmonic and bounded in R, and take continuously the boundary value zero on $C_0+\beta_1+\beta_2+\cdots+\beta_m$ and unity in the remaining boundary points of R, except that in points common to the β_j and $C_2+\cdots+C_n$ and in end points of the β_j, no continuous boundary value is required.*

If $\omega(z, C_0, R_0)$ denotes the harmonic measure of C_0 in the point z with respect to the annular region R_0, then for constant μ the largest region $\omega(z, C_0, R_0)>\mu\geqq 1/2$ which contains no points of $C_2+\cdots+C_n$ contains no critical points of $u(x, y)$.

Theorem 4 is proved by mapping onto the unit circle the universal covering surface of R_0, and by applying a slight generalization of Theorem 2. We omit the proof.

We turn now to a generalization of Theorems 2 and 4, in a more general situation. Let R be a region bounded by the whole or part of the mutually disjoint Jordan curves, $C_1, C_2, \cdots, C_k$ (which together bound a region R_0) and by mutually disjoint Jordan arcs or curves $C_{k+1}, \cdots, C_n$ in the closure of R_0; some or all of the latter arcs or curves may have points in common with $C_1+C_2+\cdots+C_k$. Let a finite number of arcs $\alpha_1, \alpha_2, \cdots, \alpha_m$ of $C_1+C_2+\cdots+C_k$ belong to the boundary of R and be mutually disjoint. Let the function $u(x, y)$ be harmonic and bounded in R, and take continuously the boundary values unity on $C_{k+1}+\cdots+C_n+\alpha_1+\cdots+\alpha_m$ and zero in the remaining boundary points of R, except that in points common to $C_1+\cdots+C_k$ and $C_{k+1}+\cdots+C_n$ and in end points of the α_j, no continuous boundary value is required. In studying the location of the critical points of $u(x, y)$, in order to apply Theorem **2** (in generalized form), it is natural to map onto the interior of the unit circle the universal covering surface of R_0. Any non-euclidean convex region in the unit circle containing all image points of the set $C_{k+1}+\cdots+C_n+\alpha_1+\cdots+\alpha_m$ contains all critical points of the transform of $u(x, y)$. But here we have a large choice; we may change the notation so that any subset of the arcs or curves $C_{k+1}, \cdots, C_n$ belongs to the set $C_1, C_2, \cdots, C_k$; each choice of the subset yields a new region R_0, a new conformal map, a new noneuclidean geometry, a new application of Theorem 2 (generalized), and a new conclusion.

Throughout the present note we have studied in detail harmonic functions which for a simply connected region R_0 take on the values zero (on arcs of the boundary of R_0) and unity (on arcs of the boundary or curves in R_0). By the same methods one can also study harmonic functions which take on the values zero (on arcs of the boundary of R_0), unity (on arcs of the boundary or curves in R_0), and minus unity (on arcs of the boundary or curves in R_0); the results generalize those previously obtained by the writer (loc. cit.) and can be still further generalized to regions of higher connectivity by a conformal map of the universal covering surfaces of such regions.

Harvard University