NOTE ON THE LOCATION OF THE CRITICAL POINTS
OF HARMONIC FUNCTIONS

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By a limiting process, a theorem recently proved by the writer can be generalized, and yields a new result with interesting applications which we wish to record here. We take as point of departure the following theorem.

THEOREM 1. Let the region $R$ of the extended $(x, y)$-plane be bounded by a finite number of mutually disjoint Jordan curves $C_0, C_1, C_2, \ldots, C_n$. Let the function $u(x, y)$ be harmonic in $R$, continuous in the corresponding closed region, equal to zero on $C_0$ and to unity on $C_1, C_2, \ldots, C_n$. Denote by $R_0$ the region bounded by $C_0$ containing the curves $C_1, C_2, \ldots, C_n$ in its interior; define noneuclidean straight lines in $R_0$ as the images of arcs of circles orthogonal to the unit circle, when $R_0$ is mapped conformally onto the interior of the unit circle.

If $\Pi$ is any non-euclidean convex region in $R_0$ which contains all the curves $C_1, C_2, \ldots, C_n$, then $\Pi$ also contains all critical points of $u(x, y)$ in $R$.

We extend Theorem 1 by admitting arcs of $C_0$ on which $u(x, y)$ is prescribed to take the value unity, and also by admitting the intersection of curves $C_1, C_2, \ldots, C_n$ with $C_0$:

THEOREM 2. Let the region $R$ be bounded by the whole or part of the Jordan curve $C_0$, and by mutually disjoint Jordan arcs or curves $C_1, C_2, \ldots, C_n$ in the closed interior of $C_0$; some or all of the latter arcs or curves may have points in common with $C_0$. Let a finite number of arcs $\alpha_1, \alpha_2, \ldots, \alpha_m$ of $C_0$ belong to the boundary of $R$ and be mutually disjoint. Let the function $u(x, y)$ be harmonic and bounded in $R$, and take continuously the boundary values unity on $C_1, C_2, \ldots, C_n$, $\alpha_1, \alpha_2, \ldots, \alpha_m$ and zero in the remaining boundary points of $R$, except that in points common to $C_0$ and $C_1+C_2+\ldots+C_n$ and in end points of the $\alpha_j$, no continuous boundary value is required. Denote by $R_0$ the region bounded by $C_0$ containing $R$, and define noneuclidean straight lines in $R_0$ by mapping $R_0$ onto the interior of a circle. If $\Pi$ is any closed region in the closure of $R_0$ which is non-euclidean convex and which contains $C_1+C_2+\ldots+C_n+\alpha_1+\alpha_2+\ldots+\alpha_m$, then $\Pi$ contains all critical points of $u(x, y)$ in $R$.

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Theorem 2 may be proved by mapping $R_0$ onto the interior of the unit circle; we retain the original notation. The region $R$ can be approximated by a region $R'$ bounded by $C_0$ and by Jordan curves $C_1', C_2', \ldots, C_n'$, $\alpha_1', \alpha_2', \ldots, \alpha_m'$ in $R_0$ which are mutually disjoint and disjoint with $C_0$ and which respectively approximate $C_1, C_2, \ldots, C_n, \alpha_1, \alpha_2, \ldots, \alpha_m$. Let the function $u'(x, y)$ be harmonic in $R'$, continuous in the corresponding closed region, zero on $C_0$ and unity elsewhere on the boundary of $R'$. Then as $R'$ suitably approaches $R$, the variable function $u'(x, y)$ approaches $u(x, y)$ throughout $R$, uniformly on any closed set interior to $R$; we omit the proof. Any critical point of $u(x, y)$ interior to $R$ is a limit point of critical points of the variable function $u'(x, y)$, so Theorem 2 follows from Theorem 1.

A further general result has recently been established\textsuperscript{2} for the case $n = 0$, which constructs $\Pi$ in $R_0$ not by joining the ends of each arc of $C_0$ in the complement of the set $\alpha_j$ by a non-euclidean line but by similarly joining the ends of each double arc composed of an $\alpha_j$ plus one of the adjoining arcs of $C_0$ complementary to the set $\alpha_1 + \alpha_2 + \cdots + \alpha_m$. It is still true (we shall refer to this result as Theorem 3) that $\Pi$ contains all critical points in $R$ of the corresponding harmonic function $u(x, y)$.

Theorem 3 is more powerful than Theorem 2 for the case $n = 0$, but requires for its application essentially the use of a specific conformal map, and the latter quality may be an advantage or a disadvantage. It is an indication of the power of Theorem 2 that in the application of it to a given configuration, with or without the auxiliary use of conformal mapping, there may obviously be some arbitrariness in the notation, especially as to what shall be chosen as the region $R_0$. So far as convenience is concerned, it is desirable to choose simple configurations, where the totality or useful subset of non-euclidean lines is easily determined. It is also well to choose $R_0$ in such a way that the point set $C_1 + \cdots + C_n + \alpha_1 + \cdots + \alpha_m$ is as nearly non-euclidean convex as possible. But if the aim is precision, the larger $R_0$ the better, as we proceed to indicate in a special but typical case.

In Theorem 2, let $C_0$ be the unit circle in the $z$-plane, $n = 2$, $m = 0$, with $C_1$ and $C_2$ mutually disjoint Jordan arcs whose end points lie on $C_0$ and whose interior points lie interior to $C_0$. Let the subregion $R$ of the interior of $C_0$ be bounded by $C_1, C_2$, and two appropriate arcs of $C_0$. In the actual application of Theorem 2, we can choose $R_0$ as the interior of $C_0$, or as the region $R_1$ containing $R$ bounded by $C_1$

and a suitable arc of \( C_0 \), or as the region \( R_2 \) containing \( R \) bounded by \( C_2 \) and a suitable arc of \( C_0 \), or as \( R \). We now show as a general indication but without a complete rigorous proof that among these choices the most precise results are obtained by choosing \( R_0 \) as the interior of \( C_0 \).

Map (for instance) the region \( R \) onto the interior of the unit circle in the \( w \)-plane. Let \( \alpha_z \) be an arbitrary arc of \( C_0 \) belonging to the boundary of \( R \), which corresponds to the arc \( \alpha_w \) in the \( w \)-plane. Let \( \alpha'_z \) be the circular arc having the same end points as \( \alpha_z \), orthogonal to \( \alpha_z \), and whose interior points lie interior to \( C_0 \); we assume \( \alpha'_z \) to lie in the closure of \( R \). Let \( \alpha'_w \) be the circular arc having the same end points as \( \alpha_w \), orthogonal to \( \alpha_w \), and whose interior points lie in \( |w| < 1 \). The arcs \( \alpha'_z \) and \( \alpha'_w \) determine the respective non-euclidean geometries in the \( z \)-plane and \( w \)-plane, and it follows from a general theorem due to R. Nevanlinna\(^3\) that the region bounded by \( \alpha_z \) and \( \alpha'_z \) contains every point of the image of the region bounded by \( \alpha_w \) and \( \alpha'_w \). Corresponding to every arc \( \alpha_z \) belonging to the boundary of \( R \) and on which the prescribed boundary value of \( u(x, y) \) is zero, and to the adjacent arc \( \alpha'_z \), with no point of \( C_1 \) or \( C_2 \) in the lens-shaped region between \( \alpha_z \) and \( \alpha'_z \) there exists in the \( w \)-plane an arc \( \alpha'_w \) whose end points correspond to those of \( \alpha_z \) under the conformal map, such that the interior points of the arc \( \alpha'_w \) lie interior to the lens-shaped region bounded by \( \alpha_w \) and the image of \( \alpha'_w \). It follows that if we neglect arcs \( \alpha'_z \) that cut \( C_1 \) or \( C_2 \) in \( R \), then in this particular case Theorem 2 can be more favorably applied by choosing \( R_0 \) as the interior of \( C_0 \), that is to say, as large as possible.

The remark just made is of fairly general application. Moreover, in the specific case used, the interior of the given \( C_0 \) may be enlarged, without altering \( R \) or \( u(x, y) \), by adding to \( R_0 \) regions adjacent to the arcs \( A \) of the given \( C_0 \) bounded by the end points of \( C_1 \) and \( C_2 \), the arcs \( A \) not being part of the boundary of \( R \). Indeed, we may even adjoin an infinitely many sheeted logarithmic Riemann surface along each arc \( A \); this is equivalent to mapping onto the interior of the unit circle the region \( R \) plus auxiliary regions, so that with the omission of two exceptional points the circumference of the unit circle corresponds only to that part of the boundary of \( R \) on which the prescribed boundary value of \( u(x, y) \) is zero. The image of \( C_1 \) (and likewise of \( C_2 \)) under this map is a Jordan curve which except for a single point lies interior to the unit circle.

Still another instructive kind of conformal map can be used, namely to map \( R + C_1 + C_2 \) onto the interior of the unit circle in such a way

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\(^3\) Eindeutige analytische Funktionen, Berlin, 1936, p. 51.
that $C_1$ and $C_2$ correspond to radial slits, while the part of the boundary of $R$ on which $u(x, y)$ has the prescribed boundary value zero corresponds to the whole circumference less two points. Here the region $\Pi$ of Theorem 2 may degenerate to a line segment.

Another indication of the power of Theorem 2 is the following. Let $C_0$ be the unit circle, $n=1$ with $C_1$ a concentric circle of radius $r_1<1$; let an arc $\alpha$ (not the whole circle) of $C_0$ contain all the arcs $\alpha_j$. By a conformal map of the universal covering surface of $R$ onto the unit circle and application of Theorem 3 extended to the case of an infinite number of arcs, $\alpha_j$, it follows (loc. cit. footnote 2) that in the original plane no critical points of $u(x, y)$ lie in the annulus $r_1<r<r_1^{1/2}$; a second annulus $r_2<r<1$ free from critical points can also be determined by this method. By Theorem 2, any circle cutting $C_0$ orthogonally in two points of the complement of $\alpha$ and containing in its interior no point of $\alpha$ or of $C_1$ contains in its interior no critical point of $u(x, y)$. In all, these conclusions may leave only a very small subregion of $R$ as the portion in which the critical points of $u(x, y)$ lie.

We continue with a generalization of this result, a further application of Theorem 2:

**Theorem 4.** Let $R$ be a region bounded by the whole of the Jordan curve $C_0$, by the whole or part of the Jordan curve $C_1$ disjoint from $C_0$, and by mutually disjoint Jordan arcs or curves $C_2, C_3, \ldots, C_n$ in the closed interior of the annulus $R_0$ bounded by $C_0$ and $C_1$; some or all of the latter arcs and curves are permitted to have points in common with $C_1$, but none with $C_0$. Let a finite number of arcs $\beta_1, \beta_2, \ldots, \beta_m$ of $C_1$ be part of the boundary of $R$ and mutually disjoint. Let the function $u(x, y)$ be harmonic and bounded in $R$, and take continuously the boundary value zero on $C_0+\beta_1+\beta_2+\cdots+\beta_m$ and unity in the remaining boundary points of $R$, except that in points common to the $\beta_j$ and $C_2+\cdots+C_n$ and in end points of the $\beta_j$, no continuous boundary value is required.

If $\omega(z, C_0, R_0)$ denotes the harmonic measure of $C_0$ in the point $z$ with respect to the annular region $R_0$, then for constant $\mu$ the largest region $\omega(z, C_0, R_0) > \mu \geq 1/2$ which contains no points of $C_2+\cdots+C_n$ contains no critical points of $u(x, y)$.

Theorem 4 is proved by mapping onto the unit circle the universal covering surface of $R_0$, and by applying a slight generalization of Theorem 2. We omit the proof.

We turn now to a generalization of Theorems 2 and 4, in a more general situation. Let $R$ be a region bounded by the whole or part of
the mutually disjoint Jordan curves, $C_1, C_2, \ldots, C_k$ (which together bound a region $R_0$) and by mutually disjoint Jordan arcs or curves $C_{k+1}, \ldots, C_n$ in the closure of $R_0$; some or all of the latter arcs or curves may have points in common with $C_1+C_2+\cdots+C_k$. Let a finite number of arcs $\alpha_1, \alpha_2, \ldots, \alpha_m$ of $C_1+C_2+\cdots+C_k$ belong to the boundary of $R$ and be mutually disjoint. Let the function $u(x, y)$ be harmonic and bounded in $R$, and take continuously the boundary values unity on $C_{k+1}+\cdots+C_n+\alpha_1+\cdots+\alpha_m$ and zero in the remaining boundary points of $R$, except that in points common to $C_1+\cdots+C_k$ and $C_{k+1}+\cdots+C_n$ and in end points of the $\alpha_j$, no continuous boundary value is required. In studying the location of the critical points of $u(x, y)$, in order to apply Theorem 2 (in generalized form), it is natural to map onto the interior of the unit circle the universal covering surface of $R_0$. Any non-Euclidean convex region in the unit circle containing all image points of the set $C_{k+1}+\cdots+C_n+\alpha_1+\cdots+\alpha_m$ contains all critical points of the transform of $u(x, y)$. But here we have a large choice; we may change the notation so that any subset of the arcs or curves $C_{k+1}, \ldots, C_n$ belongs to the set $C_1, C_2, \ldots, C_k$; each choice of the subset yields a new region $R_0$, a new conformal map, a new noneuclidean geometry, a new application of Theorem 2 (generalized), and a new conclusion.

Throughout the present note we have studied in detail harmonic functions which for a simply connected region $R_0$ take on the values zero (on arcs of the boundary of $R_0$) and unity (on arcs of the boundary or curves in $R_0$). By the same methods one can also study harmonic functions which take on the values zero (on arcs of the boundary of $R_0$), unity (on arcs of the boundary or curves in $R_0$), and minus unity (on arcs of the boundary or curves in $R_0$); the results generalize those previously obtained by the writer (loc. cit.) and can be still further generalized to regions of higher connectivity by a conformal map of the universal covering surfaces of such regions.

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