BOOK REVIEWS

Die Bewegungsgruppen der Kristallographie. By J. J. Burckhardt. (Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften.) Basel, Birkhauser, 1947. 186 pp. 24.50 S. fr.; bound 29 S. fr.

The systematic study of the discrete space groups and their application to crystallography dates back to Schoenflies (1891) and Federow (1892). Bieberbach in 1910 and Frobenius, with a simpler proof, in 1911 showed that there exist only a finite number of these groups having a finite fundamental region. Treatises listing the 230 space groups and the corresponding point lattices have been written by P. Niggli, *Geometrische Kristallographie des Diskontinuums* (Leipzig, 1919) and R. W. G. Wyckoff, *The analytical expression of the results of the theory of space groups* (Carnegie Institution of Washington Publication 318, 2d ed., 1930). Other contributions to the theory of space groups have been made by D. Hilbert, C. Jordan, L. Schläfli, H. S. M. Coxeter, W. Nowacki, A. Speiser, G. Wintgen, and others to whom references are given. The purpose of the author is to develop the theory of space groups systematically with primary emphasis not on the geometric crystal classes but on the arithmetic theory of space lattices.

Chapter I opens with a careful presentation of the fundamental notions of vectors and matrices, and the particular properties of certain orthogonal and integral unimodular matrices which represent the symmetry operations of a point lattice. The point groups are orthogonal in rectangular coordinates, but integral unimodular when referred to lattice coordinates. If y, x, a are ν-dimensional column vectors, A is a ν by ν matrix and E the unit matrix, then the general rigid motion $y=Ax+a$ is denoted by (A, a). In any of the 230 space groups G, the transformations (E, a) form the invariant abelian subgroup T of translations, and the quotient group G/T is isomorphic to a finite group G_0 of integral unimodular matrices, called a crystal class.

Chapter II contains a description and classification of the crystal classes—the symmetry groups of a given point lattice which leave fixed one lattice point. Two crystal classes are considered *geometrically* equivalent if one can be transformed into the other by a *nonsingular* transformation. They are *arithmetically* equivalent only if a transforming matrix can be chosen which is *integral* and *unimodular.*

Great stress is laid on this distinction. The 14 space lattices give rise to 32 ternary geometric classes, but 73 ternary arithmetic classes. The latter fall into seven systems as follows: 2 triclinic, 6 monoclinic, 13 orthorhombic, 16 tetragonal, 15 cubic, 5 rhombohedral, and 16 hexagonal. The author derives them first from the lattices and their geometric classes in a manner which generalizes to spaces of more than three dimensions, and then gives a simpler proof of the completeness of the three-dimensional classification, based on a study of the symmetry of the fundamental cell. A lucid diagrammatical representation of the crystal classes and their subgroups is shown on p. 72. Typographical errors appear on p. 39, matrix E in (2), and on pp. 98–99 where in three equations (3) the expression $xy/2$ should read $2xy$.

Chapter III is concerned with the derivation and classification of the 230 non-equivalent space groups having finite fundamental regions. Should no distinction be made between right and left-handed screw motions, this number would reduce to 219. Two space groups are to be considered equivalent only if they belong to the same arithmetic class G_0. If, referred to lattice coordinates, the elements of G_0 are the integral unimodular matrices A_i, and if representatives of the n cosets of the space groups G with respect to T (subgroup of translations) are (A_i, a_i), then according to Frobenius a choice of origin for which $\sum a_i \equiv 0 \pmod 1$ will imply that na_i are integral vectors whose components can be chosen to lie between 0 and $n-1$. For each arithmetic class there is always the null solution $a_i = 0$, and there may be at most a finite number of other solutions of the Frobenius congruence $A_i a_k + a_i \equiv a_l$ (where $A_i A_k = A_l$). For a given point group G_0 with matrices A_i the two space groups given by (A_i, a_i) and (A_i, b_i) are called equivalent solutions if and only if the congruences $a_i - b_i \equiv (E - A_i)s$ have a common solution s for each i. It is clear then that if G_0 is reducible and contains the identical representation as a component, and if the corresponding component of $a_i - b_i$ is $\not\equiv 0$, then the two space groups (A_i, a_i) and (A_i, b_i) are not equivalent. For example, the two plane crystal classes C_2 and C_s are generated respectively by diag $(-1, -1)$ and diag $(1, -1)$. The former gives rise to but one group of motions (the null solution) but the latter admits also a second group containing a sliding reflection whose translation vector is either $(1/2, 0)$ or $(1/2, 1/2)$. Following the complete derivation and classification of the 17 groups of motions of the plane (with finite fundamental region) arising from 13 arithmetic classes of which all except C_s, C_{2v} and C_{4v} contribute only the null solution, a few simple space groups are derived immediately by adjoining the identity representation to G_0. The remaining space groups are then derived in detail and classified within the rhombohedral, hexagonal, monoclinic, rhombic, tetragonal, and cubic systems.

The two final sections are devoted to the study of special families of space groups in n dimensions, such as those arising from the cyclic, symmetric, and alternating groups on n symbols.

The book is clearly written and self-contained, except in the section beginning on p. 91 where the ternary arithmetic classes are listed. Here the reader without previous knowledge of the notations of crystallography may have some difficulty reading the rather condensed summary of the 73 ternary arithmetic classes. The groups of motions in the plane are illustrated by excellent figures, but no attempt is made to illustrate the 230 space groups by drawings such as are given by Wyckoff. The emphasis in the book is clearly on the mathematical derivation rather than the pictorial representation of the 230 space groups.

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Methods of mathematical physics. By Harold Jeffreys and Bertha S. Jeffreys. New York, Macmillan; Cambridge University Press, 1946. 9+679 pp. $15.00.

The book starts with a substantial chapter on real variable—Dedekind sections, sequences, series, continuity, integration, mean value theorems. Chapters 2, 3, and 4 cover vectors, cartesian tensors, and matrices, and these are followed by chapters on multiple integrals and potential theory. Operational methods and their applications occupy two chapters, and a long chapter is devoted to numerical methods. A short chapter on calculus of variations brings us to what may be regarded as the mid-point of the book, attained almost entirely without the use of complex numbers.

The essential elements of the theory of functions of a complex variable are covered in two chapters. This opens up a wide field, and chapters follow on conformal representation, Fourier's theorem, factorial (gamma) functions, linear differential equations of the second order, asymptotic expansions, equations of wave motion and heat conduction (three chapters), Bessel functions and applications, confluent hypergeometric functions, Legendre functions, elliptic functions. The book ends with explanatory notes, an appendix on notation, and an index.

Each chapter has a set of examples, a stimulating collection culled from examinations of the Universities of Cambridge, London, and Manchester.