ON RINGS OF ANALYTIC FUNCTIONS

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Let $D$ be a domain in the complex plane (Riemann sphere) and $R(D)$ the totality of one-valued regular analytic functions defined in $D$. With the usual definitions of addition and multiplication $R(D)$ becomes a commutative ring (in fact, a domain of integrity). A one-to-one conformal transformation $\zeta = \phi(z)$ of $D$ onto a domain $\Delta$ induces an isomorphism $f \mapsto f^*$ between $R(D)$ and $R(\Delta): f(z) = f^*[\phi(z)]$. An anti-conformal transformation

$$\zeta = \overline{\phi(z)}$$

also induces an isomorphism:

$$\overline{f(z)} = f^*[\phi(z)].$$

The purpose of this note is to prove the converses of these statements.

**Theorem I.** If $R(D)$ is isomorphic to $R(\Delta)$, then there exists either a conformal or an anti-conformal transformation which maps $D$ onto $\Delta$.

**Theorem II.** If $D$ and $\Delta$ possess boundary points, then every isomorphism between $R(D)$ and $R(\Delta)$ is induced by a conformal or an anti-conformal transformation of $D$ onto $\Delta$.

Theorem I may be regarded as a complex variable analogue of theorems characterizing a topological space in terms of the family of its continuous functions. If $R(D)$ is made into a topological ring by defining $f_n \to f$ to mean that $f_n(z) \to f(z)$ uniformly in every bounded closed subset of $D$, then Theorem II implies that, except for a trivial special case, every isomorphism between $R(D)$ and $R(\Delta)$ is of necessity a homeomorphism.

To prove the theorems we consider a fixed isomorphism between $R(D)$ and $R(\Delta)$. It takes a function $f(z), z \in D$, into a function $f^*(\zeta), \zeta \in \Delta$, a set $S \subset R(D)$ into a set $S^* \subset R(\Delta)$. Let $c$ be a complex constant.

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1 After this paper was completed the author learned about a closely related unpublished result which was obtained by C. Chevalley and S. Kakutani several years ago. Chevalley and Kakutani proved that if to each boundary point $W$ of $B$ there exists a bounded analytic function defined in $B$ and possessing at $W$ a singularity then $B$ is determined (modulo a conformal transformation) by the ring of all bounded analytic functions. The author is indebted to Professor Chevalley for the opportunity of reading a draft of the paper containing the proof.

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For the sake of brevity we denote the element of $R(D)$ corresponding to the functions $f(z) \equiv c$ (or the element of $R(\Delta)$ corresponding to the function $g(\xi) \equiv c$) by the letter $c$. We call a complex number rational if its real and imaginary parts are rational.

**Lemma 1.** Either $i^* = i$ and for every rational complex constant $r: r^* = r$, or $i^* = -i$ and $r^* = \bar{r}$.

The proof is clear.

**Lemma 2.** If $c$ is a constant, so is $c^*$.

**Proof.** If $c$ is rational the assertion is contained in the preceding lemma. Irrational constants $c$ are characterized by the existence of the inverse of the element $c - r$ for every rational constant $r$.

**Lemma 3.** All elements of $R(D)$ are constants if and only if $D$ is the whole complex plane including the point at infinity.

The proof is clear.

Lemmas 2 and 3 contain the proof of Theorem I for the case when $D$ is the domain $0 \leq |z| \leq \infty$. In what follows we consider only domains possessing boundary points. Without loss of generality we assume that neither $D$ nor $\Delta$ contains the point at infinity. We also assume that $i^* = i$; the case $i^* = -i$ can be treated in the same way.

We denote the set of all functions belonging to $R(D)$ and vanishing at a point $a \in D$ by $I_a$. The set $I_a \subset R(\Delta), \alpha \in \Delta$, is defined similarly.

**Lemma 4.** There exists a one-to-one mapping $z \rightarrow z' = \phi(z)$ of $D$ onto $\Delta$ such that $I_{a^*} = I_{\phi(a)}$.

**Proof.** Every element of $R(D)$ generates a principal ideal $(f)$, that is, the set of all elements of the form $fh, h \in R(D)$. $(f)$ is said to be a maximal principal ideal if $(f) \neq R(D)$ and if $(f) \subset (g) \neq R(D)$ implies that $(f) = (g)$. It is clear that $(f)^* = (f^*)$ and that $(f^*)$ is a m.p. ideal if and only if $(f)$ is. Hence Lemma 4 is an immediate consequence of the following lemma.

**Lemma 5.** $(f)$ is a m.p. ideal if and only if $(f) = I_a$.

**Proof.** $I_a$ is the principal ideal generated by the function $z-a$. If $I_a \subset (g) \neq R(D)$, then $g(z)$ must possess zeros in $D$. Since $z-a = g(z)h(z), h \in R(D), g(a) = 0$ and $g \in I_a$. On the other hand, if $f(z)$ has no zeros in $D$, then $(f) = R(D)$, and if $f(a) = f'(a) = 0$, or if $f(a) = f(b) = 0, a \neq b$, then $(f)$ is contained in and different from the
principal ideal generated by the function \( z - a \). It follows that \((f)\) is a m.p. ideal if and only if \( f(z) = (z - a)e^{h(z)}, a \in D, h \in R(D) \).

**Lemma 6.** For every point \( z_0 \in D, f(z_0)^* = f^* [\phi(z_0)] \).

**Proof.** If \( c \) is a constant such that \( f(z_0) = c \), then \( c - f \) belongs to \( I_{z_0} \), so that \( c^* - f^* \) belongs to \( I_{\phi(z_0)} \) and \( f^* [\phi(z_0)] = c^* \).

The following two lemmas are immediate consequences of Lemma 6.

**Lemma 7.** If \( z_0 \in D \) and \( f(z_0) \) is a rational number, then \( f(z_0) = f^* [\phi(z_0)] \).

**Lemma 8.** If \( f(z) \) is univalent in \( D \), then \( f^*(\zeta) \) is univalent in \( \Delta \).

**Lemma 9.** Let \( f(z) \) be a univalent function defined in \( D \), let \( f(D) \) be the image of \( D \) under the transformation \( w = f(z) \), and let \( W \) be the (finite) limit of a convergent sequence of distinct rational points \( \{w_n\} \) belonging to \( f(D) \). \( W \) is a boundary point of \( f(D) \) if and only if there exists a function \( g(z) \in R(D) \) such that \( g[h(w_n)] = n \), \( h(w) \) being the function inverse to \( w = f(z) \).

**Proof.** If \( W \) is a boundary point of \( f(D) \), choose an entire function \( F(Z) \) such that \( F[(W - w_n)^{-1}] = n \). The function \( g(z) = F\{ (W - f(z))^{-1} \} \) satisfies the conditions of the lemma. On the other hand, if \( W \) is an interior point of \( f(D) \), \( W = f(a), a \in D, \) and for every \( g(z) \in R(D) \), \( \lim g[h(w_n)] \) exists and is finite.

**Lemma 10.** Let \( f(z) \) be a univalent function defined in \( D \), so that \( f^*(\zeta) \) is a univalent function defined in \( \Delta \). The domains \( f(D) \) and \( f^*(\Delta) \) are identical.

**Proof.** It follows from Lemma 7 that the rational points belonging to \( f(D) \) are identical with the rational points belonging to \( f^*(\Delta) \). \( f(D) \) is the set of all limit points of its rational points, except those limit points which lie on the boundary of \( f(D) \). A similar remark applies to \( f^*(\Delta) \). But Lemmas 7 and 9 imply that if a sequence of rational points from \( f(D) \) converges to a boundary point \( W, W \) is a boundary point of \( f^*(\Delta) \).

Lemma 10 contains the proof of Theorem I for domains possessing boundary points.

The proof of Theorem II depends on the following lemma.

**Lemma 11.** If \( D \) possesses boundary points, then \( c^* = c \) for every constant \( c \).

We prove this lemma in several steps. Let \( B \) be any domain. By
For every univalent function \( f \in R(D) \) and for every constant \( c \) the difference \( c - c^* \) belongs to \( m[f(D)] \).

**Proof.** The function \( f_1 = f + c \) is univalent in \( D \), and the functions \( f^* \) and \( f_1^* = f^* + c^* \) are univalent in \( \Delta \) (Lemma 8). By virtue of Lemma 10, \( f(D) \) is identical with \( f^*(\Delta) \), and \( f_1(D) \) is identical with \( f_1^*(\Delta) \). But a translation by \( c \) takes \( f(D) \) into \( f_1(D) \) and a translation by \( -c^* \) takes \( f_1^*(\Delta) \) into \( f^*(\Delta) \). Thus the translation by \( c - c^* \) leaves \( f(D) \) invariant.

**Lemma 13.** If \( D \) possesses finite boundary points, then there exists a univalent function \( f \in R(D) \) such that \( m[f(D)] \) is a discrete set.

**Proof.** Let \( f \) be a fixed univalent function defined in \( D \). The set \( m[f(D)] \) is closed and a modul, that is, it contains \( d_1 - d_2 \) whenever it contains \( d_1 \) and \( d_2 \). Assume that \( m[f(D)] \) is not discrete. Then it either contains all points, or all points of a straight line. If \( W \) is a finite boundary point of \( f(D) \), every point \( W + d, d \in m[f(D)] \), is a boundary point. It follows that the boundary of \( f(D) \) contains a finite straight segment \( S \). Let \( Z(w) \) be the function which maps the domain exterior to \( S \) conformally onto \( |Z| < 1 \). The function \( g(z) = Z[f(z)] \) is univalent and bounded in \( D \). It follows that \( m[g(D)] \) contains only the point 0.

Now we can prove Lemma 11 under the hypothesis of Lemma 13. Let \( f \in R(D) \) be univalent and such that \( m[f(D)] \) is discrete. For every positive number \( t \) set \( f_t = tf \). By Lemma 12 the difference \( c - c^* \) belongs to \( m[f_t(D)] \), that is, the number \( (c - c^*)/t \) belongs to \( m[f(D)] \). It follows that \( c - c^* = 0 \).

It remains to establish Lemma 11 for the case of the whole finite plane.

**Lemma 14.** If \( D \) is the domain \( |z| < \infty \) and \( \Delta \) the domain \( |\xi| < \infty \), then \( c_n \to \infty \) implies \( c_n^* \to \infty \).

**Proof.** Set \( f(z) = z \). Then \( f^*(\xi) \) is univalent in \( \Delta \), that is, \( f^*(\xi) = A\xi + B \), \( A, B \) = const., \( A \neq 0 \). For this \( f \) and for \( z_0 = c \), Lemma 6 yields \( c^* = A\phi(c) + B \). Hence \( c_n^* \to \infty \) whenever \( \phi(c_n) \to \infty \). But \( c_n \to \infty \).

* An alternative argument was suggested to the author by C. Loewner. Assume that \( c - c^* = re^{i\theta} \), \( r \neq 0 \). It is easy to see that for any univalent \( f \) and for any domain \( D \) satisfying the hypothesis of Lemma 13 there exists a (finite or semi-infinite) straight segment \( S \) whose interior points belong to \( f(D) \) and whose end points are boundary points of \( f(D) \). By a linear transformation we can achieve that \( S \) be the semi-infinite segment \( z = le^{i\theta} \), \( l > 0 \). It follows that \( c - c^* \) does not belong to \( m[f(D)] \).
implies the existence of a function \( g \in \mathcal{R}(D) \) such that \( g(c_n) \) is rational and \( g(c_n) \to \infty \). By Lemma 7, \( g^*[\phi(c_n)] \to \infty \), from which it follows that \( \phi(c_n) \to \infty \).

In order to prove Lemma 11 under the hypothesis of Lemma 14 we note that the transformation \( c \to c^* \) is an automorphism of the complex field. This automorphism is continuous by virtue of Lemma 14. Hence \( c^* = c \), for we assumed that \( i^* = i \).

Lemma 11 being established, Lemma 6 yields the following lemma.

**Lemma 15.** If \( D \) possesses boundary points, then \( f(z_0) = f^*[\phi(z_0)] \) for every \( z_0 \in D \).

This lemma would contain Theorem II if we would know that \( \phi(z) \) is analytic in \( D \). To show this we select for \( f \) the function \( f(z) = z \). Lemma 15 shows that \( \phi \) is the function inverse to the univalent (analytic) function \( f^* \).

In the statement of Theorem I the ring of all analytic functions cannot be replaced by the subring \( B(D) \) of all bounded analytic functions defined in \( D \), even if \( B(D) \) is treated as a normed ring (with \( \|f\| = \text{l.u.b. } |f(z)| \), and the isomorphism between \( B(D) \) and \( B(\Delta) \) is required to be norm preserving. In fact, let \( D \) be the domain \( |z| < 1 \), and let \( \Delta \) be the domain \( 0 < |z| < 1 \). The normed rings \( B(D) \) and \( B(\Delta) \) are identical.

Neither is it possible to replace \( R(D) \) by the linear space \( L(D) \) of all analytic functions defined in \( D \), even if \( L(D) \) is considered to be a topological space (with the topology defined above) and the isomorphism between \( L(D) \) and \( L(\Delta) \) is required to be a homeomorphism.

In fact, let \( D \) be the domain \( 0 < r < |z| < 1 \), and let \( D_1 \) and \( D_2 \) denote the domains \( |z| < 1 \) and \( |z| > r \), respectively. Every function \( f \in L(D) \) admits a unique Laurent decomposition: \( f(z) = g(z) + \frac{h(z)}{z} \), \( g \in L(D_1) \), \( h \in L(D_2) \). It is easy to see that \( f_n \to f \) if and only if \( g_n \to g \) and \( h_n \to h \). On the other hand, the linear subspace of \( L(D_2) \) consisting of all functions of the form \( h(z)/z \), \( h \in L(D_2) \), is topologically isomorphic to \( L(D_2) \), and \( L(D_2) \) is topologically isomorphic to \( L(D_1) \). It follows that \( L(D) \) is topologically isomorphic to the direct sum of two spaces \( L(D_1) \). Thus \( L(D) \) considered as an abstract linear topological space is independent of \( r \).

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\(^3\) As a matter of fact, every isomorphism is.

\(^4\) Cf., however, footnote 1.