# AN EXTENSION OF ALEXANDROFF'S MAPPING THEOREM

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**1. Introduction.** Alexandroff's fundamental mapping theorem (Überführungssatz) is a basic tool of combinatorial topology. By its use, many mapping theorems for rather general topological spaces can be shown to be consequences of the corresponding theorems for polytopes. It is especially useful in proving imbedding theorems and approximation theorems.

The chief purpose of this paper is to determine the precise conditions under which this fundamental theorem holds. It will be shown that the theorem holds in full generality, that is, for all coverings, if and only if the space is both paracompact and normal. If the space is normal, the theorem holds for all coverings which have locally finite refinements and for no others.

**2. Terminology.** The mapping theorem of Alexandroff concerns mappings of a space into the nerve of a covering.

By a *mapping* we mean a continuous transformation. By a *space* we mean a topological space, in general not satisfying any separation axiom. By a *covering* we mean a covering of the space by a finite or infinite collection of open sets.

The *nerve* of a covering $\mathfrak{U}$: $\{U_\alpha\}$ is a simplicial polytope, with vertices $u_\alpha$ in 1–1 correspondence with the nonempty sets $U_\alpha$ of the covering, such that $u_\alpha, u_\beta, \cdots, u_\gamma$ are vertices of a simplex of the nerve if and only if the corresponding sets $U_\alpha, U_\beta, \cdots, U_\gamma$ have a common point. We assume that the nerve is realized as a topological space in one of the following ways. The *natural nerve* $N(\mathfrak{U})$ is the nerve realized with the natural metric: $\rho(x, y) = (\sum(x_\alpha - y_\alpha)^2)^{1/2}$, where $x_\alpha$, $y_\alpha$ are barycentric coordinates of $x$ and $y$. The *geometric nerve* $G(\mathfrak{U})$ is the nerve realized with the geometric topology of Lefschetz [10, p. 9]:[1] the stars of the vertices of repeated regular subdivisions form a basis for the open sets of $G(\mathfrak{U})$. It is known [12] that $G(\mathfrak{U})$ is a metrizable space. The natural and geometric topologies coincide if and only if the nerve is locally of finite dimension [11, footnote 4].

Following Dieudonné, we call a covering $\mathfrak{U}$ of a space $R$ *locally finite*[2]

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] Locally finite = neighborhood-finite. "Locally finite" has been used by Lefschetz to mean star-finite.

if each point of $R$ has a neighborhood meeting only a finite number of sets of $\mathfrak{U}$. A covering $\mathfrak{V}$ is called a refinement of $\mathfrak{U}$ if every set of $\mathfrak{V}$ is contained in some set of $\mathfrak{U}$. A space $R$ is called *paracompact* (see [4]) if every covering of $R$ has a locally finite refinement. A covering of a space $R$ is called *point-finite* if each point of $R$ is contained in only a finite number of sets of the covering. Clearly, every locally finite covering is point-finite.

A mapping $f$ of a space $R$ into the nerve of a covering $\mathfrak{U}$ is called *canonical* with respect to $\mathfrak{U}$ if the inverse image of the star[3] of each vertex of the nerve is contained in the corresponding set of $\mathfrak{U}$; in symbols: $f^{-1}(\mathrm{St}u_\alpha) \subset U_\alpha$. If $\mathfrak{U}$ is point-finite, the finite collection of sets $U_\alpha$ containing a point $p$ of $R$ correspond to the vertices $u_\alpha$ of a simplex in the nerve which we call the simplex $\sigma(p)$ determined by $p$. It can be shown [5, p. 202] that a mapping $f$ of $R$ into the nerve of a point-finite covering $\mathfrak{U}$ is canonical with respect to $\mathfrak{U}$ if and only if each point $p$ of $R$ is mapped into the closure of the simplex determined by $p$, that is, $f(p) \in \bar{\sigma}(p)$. Another equivalent formulation in the case of point-finite coverings is given by Lefschetz [10, p. 40].

3. **Sufficient conditions.** The mapping theorem of Alexandroff, as modified by Kuratowski and Lefschetz, states the existence, under certain conditions, of canonical mappings of a space into the nerve of a covering.

(a) *If $\mathfrak{U}$ is a locally finite covering of a normal space $R$ there is a canonical mapping of $R$ into the natural nerve of $\mathfrak{U}$.*

This form of Alexandroff's theorem is proved in [5, Theorem 1.1]. Alternatively, the proof by Lefschetz [10, pp. 45–46] for star-finite coverings can easily be extended to locally finite coverings by using Dieudonné's theorem [4, Theorem 6] that every point-finite covering $\{U_\alpha\}$ of a normal space can be shrunk to a covering $\{V_\alpha\}$ such that, for each $\alpha$, the closure of $V_\alpha$ is contained in $U_\alpha$.

(b) *If a covering $\mathfrak{U}$ of a normal space $R$ has a locally finite refinement there is a canonical mapping of $R$ into the natural nerve of $\mathfrak{U}$.*

PROOF. Let $\mathfrak{V}: \{V_\beta\}$ be a locally finite refinement of $\mathfrak{U}: \{U_\alpha\}$. For each $V_\beta$ choose one of the sets $U_\alpha$ such that $V_\beta \subset U_\alpha$. Each $V_\beta$ thus corresponds to a unique set $U_\alpha$ containing it. Let $W_\alpha$ be the union of all the sets of $\mathfrak{V}$ which correspond to $U_\alpha$. Each $V_\beta$ is in some $W_\alpha$; hence $\mathfrak{W}: \{W_\alpha\}$ is a covering of $R$. Since $\mathfrak{V}$ is locally finite, so is $\mathfrak{W}$. If $(w_\alpha, w_\beta, \cdots, w_\gamma)$ is a simplex of $N(\mathfrak{W})$, there is a point

$$p \in W_\alpha W_\beta \cdots W_\gamma \subset U_\alpha U_\beta \cdots U_\gamma,$$

[3] The star of a vertex $v$ is the union of all simplexes having $v$ as a vertex. The star of a vertex is an open set in either topology.

and hence $(u_\alpha, u_\beta, \cdots, u_\gamma)$ is a simplex of $N(\mathfrak{U})$. Thus there is a simplicial mapping $\pi$ of $N(\mathfrak{W})$ into $N(\mathfrak{U})$ which maps each vertex $w_\alpha$ into the corresponding vertex $u_\alpha$. Clearly $\pi$ is a 1–1 mapping of $N(\mathfrak{W})$ onto a subpolytope of $N(\mathfrak{U})$. If $w_\alpha$ exists, $\pi^{-1}\mathrm{St}u_\alpha = \mathrm{St}w_\alpha$; otherwise $\pi^{-1}\mathrm{St}u_\alpha = 0$. By (a) there is a canonical mapping $g$ of $R$ into $N(\mathfrak{W})$. Then, if $f = \pi g$, $f$ is a mapping of $R$ into $N(\mathfrak{U})$. If $w_\alpha$ exists,

$$f^{-1}\ \mathrm{St}\ u_\alpha = g^{-1}\pi^{-1}\ \mathrm{St}\ u_\alpha = g^{-1}\ \mathrm{St}\ w_\alpha \subset W_\alpha \subset U_\alpha.$$

Otherwise, $f^{-1}$ St $u_\alpha = 0 \subset U_\alpha$. Hence $f$ is a canonical mapping of $R$ into $N(\mathfrak{U})$.

(c) *If $\mathfrak{U}$ is an arbitrary covering of a paracompact normal space $R$, there is a canonical mapping of $R$ into the natural nerve of $\mathfrak{U}$.*

Proof. Since $R$ is paracompact, $\mathfrak{U}$ has a locally finite refinement. Hence, by (b), a canonical mapping exists.

**4. Necessary conditions; main theorems.** We pass now to consideration of some necessary conditions for the existence of canonical mappings.

(d) *If there exists a canonical mapping $f$ of a space $R$ into the natural nerve of a covering $\mathfrak{U}$, then $\mathfrak{U}$ has a locally finite refinement.*

Proof. The natural complex $N(\mathfrak{U})$ can be shown to be a paracompact normal (metric) space [6, Theorem 2]. Hence, the covering $\{\mathrm{St}\ u_\alpha\}$ of $N(\mathfrak{U})$ by the stars of its vertices has a locally finite refinement $\{V_\beta\}$. Then $\{f^{-1}V_\beta\}$ is a locally finite covering of $R$. Since $V_\beta$ is contained in some St $u_\alpha$, $f^{-1}V_\beta \subset f^{-1}$ St $u_\alpha \subset U_\alpha$. Hence $\{f^{-1}V_\beta\}$ is a refinement of $\mathfrak{U}$.

(e) *Let $R$ be a space such that for every covering $\mathfrak{U}$ of $R$ there is a canonical mapping of $R$ into the natural nerve of $\mathfrak{U}$. Then $R$ is a paracompact normal space.*

Proof. By (d), the existence of a canonical mapping into $N(\mathfrak{U})$ implies that $\mathfrak{U}$ has a locally finite refinement. Hence every covering $\mathfrak{U}$ of $R$ has a locally finite refinement, that is, $R$ is paracompact. Since, in particular, there is a canonical mapping of $R$ into the nerve of each covering by two open sets, $R$ is normal (see [10, p. 45]).

Theorem 1. *There exists a canonical mapping of a normal space $R$ into the natural nerve of a covering $\mathfrak{U}$ if and only if $\mathfrak{U}$ has a locally finite refinement.*

Proof. This follows immediately from (b) and (d).

Theorem 2. *There exists a canonical mapping of a space $R$ into the natural nerve of an arbitrary covering if and only if $R$ is paracompact and normal.*

PROOF. This follows immediately from (c) and (e).

5. **Dimension.** Alexandroff's original theorem was partly, and indeed mainly, a theorem of dimension theory. The dimension-theoretic content of the extended (and modified) theorem is indicated by Theorem 3.

The *dimension* of a normal space $R$, dim $R$, is defined as follows: dim $R \leqq n$ means that every finite covering of $R$ has a finite refinement of order not greater than $n+1$.

Let $K$ be a simplicial polytope. Let $\sigma^n$ be an $n$-simplex of $K$, $\bar{\sigma}^n$ its closure, and $S^{n-1}$ its boundary. If $f$ maps $R$ into $K$, let $A = f^{-1}S^{n-1}$ and let $f|A$ be the mapping $f$ restricted to $A$. If $f|A$ cannot be extended to a mapping of $f^{-1}\bar{\sigma}^n$ into $S^{n-1}$, the mapping $f$ of $R$ into $K$ is called *essential* in the closed simplex $\bar{\sigma}^n$. Thus, if $f$ is essential in $\bar{\sigma}^n$, $f|A$ cannot be extended to a mapping of $R$ ($\supset f^{-1}\bar{\sigma}^n$) into $S^{n-1}$. It can be seen that a mapping $f$ of $R$ into $K$ is essential in $\bar{\sigma}^n$ if and only if, for some subset $B$ of $R$, $f|B$ is essential in $\bar{\sigma}^n$ and $f(B) \subset \bar{\sigma}^n$.

THEOREM 3. *If $\mathfrak{U}$ is a covering of a normal space $R$ and if any canonical mapping of $R$ into $N(\mathfrak{U})$ exists, then, for some canonical mapping $f$ of $R$ into $N(\mathfrak{U})$, the image $f(R)$ is a subpolytope $K$ of $N(\mathfrak{U})$,* dim $K \leqq$ dim $R$, *and $f$ is essential in every closed simplex of $K$.*

PROOF. By (d), the existence of a canonical mapping of $R$ into $N(\mathfrak{U})$ implies that $\mathfrak{U}$ has a locally finite refinement $\mathfrak{V}$. It can be shown [5, Lemma 3.4] that every locally finite covering $\mathfrak{V}$ of a normal space $R$ has a locally finite refinement $\mathfrak{W}: \{W_\beta\}$ such that there is a mapping $\phi$ of $R$ onto $N(\mathfrak{W})$ with the following properties: $\phi$ is essential in every closed simplex of $N(\mathfrak{W})$ and $\phi^{-1}$ St $w_\beta = W_\beta$ for every $\beta$.

Since $\mathfrak{W}$ is a refinement of $\mathfrak{U}$ there exists a simplicial transformation[4] $\pi$ of $N(\mathfrak{W})$ into $N(\mathfrak{U})$ which maps $w_\beta$ into $u_\alpha$ where $U_\alpha$ is a set of $\mathfrak{U}$ containing $W_\beta$. We may assume that $\pi$ is continuous on each finite subpolytope of $N(\mathfrak{W})$. Since $\mathfrak{W}$ is locally finite, each point $p \in R$ has a neighborhood $U$ meeting only a finite number of sets of $\mathfrak{W}$, and hence $\phi(U)$ is contained in a finite subpolytope of $N(\mathfrak{W})$. Then $\pi\phi$ maps a neighborhood of an arbitrary point $p$ continuously. Hence $\pi\phi$ is continuous. Let $f = \pi\phi$.

If $\pi\phi(p) \in$ St $u_\alpha$, then $\phi(p) \in$ St $w_\beta$ for some $w_\beta$ mapped into $u_\alpha$. Hence, $p \in \phi^{-1}$ St $w_\beta = W_\beta \subset U_\alpha$. Thus $f^{-1}$ St $u_\alpha \subset U_\alpha$. Therefore, $f$ is a canonical mapping.

Let $K$ be the image polytope $\pi N(\mathfrak{W}) \subset N(\mathfrak{U})$. Since $\phi(R) = N(\mathfrak{W})$, $f(R) = \pi\phi(R) = \pi N(\mathfrak{W}) = K$. Thus $f$ is a mapping of $R$ onto $K$.

[4] See, for example [5, footnote 13].

Let $\sigma^n$ be an $n$-simplex of $K$. Since $\pi N(\mathfrak{W}) = K$, there is some simplex $\sigma^m$ of $N(\mathfrak{W})$ such that $\pi\sigma^m = \sigma^n$. There is some $n$-face $\sigma_1^n$ of $\sigma^m$ which is mapped onto $\sigma^n$. (If $m = n$, $\sigma_1^n = \sigma^m$.) Since $\phi$ is essential in every closed simplex of $N(\mathfrak{W})$, $\phi$ is essential in $\bar{\sigma}_1^n$ and hence there is a subset $B$ of $R$ such that $\phi(B) \subset \bar{\sigma}_1^n$ and $\phi|B$ is essential in $\bar{\sigma}_1^n$. Since $\pi$ maps $\bar{\sigma}_1^n$ homeomorphically onto $\bar{\sigma}^n$, $\pi\phi|B$ is essential in $\bar{\sigma}^n$. Hence $\pi\phi$ is essential in $\bar{\sigma}^n$. It follows [5, Corollary 3.6] that $\dim R \geqq n$; hence each simplex of $K$ has dimension not greater than $\dim R$. Thus $f = \pi\phi$ is essential in each closed simplex of $K$ and $\dim K \leqq \dim R$. This completes the proof.

**6. Lefschetz' geometric nerve.** It will now be shown that in any theorem on canonical mappings the geometric nerve can be replaced by the natural nerve, or the natural by the geometric.

THEOREM 4. *Let $\mathfrak{U}$ be a covering of a space $R$. There exists a canonical mapping of $R$ into the geometric nerve $G(\mathfrak{U})$ if and only if there is a canonical mapping of $R$ into the natural nerve $N(\mathfrak{U})$.*

PROOF. First, let $f$ be a canonical mapping of $R$ into $N(\mathfrak{U})$. Let the vertices of $N(\mathfrak{U})$ be $\{u_\alpha\}$ and let those of $G(\mathfrak{U})$ be $\{a_\alpha\}$. It is known [10, p. 21, (9.9)] that the barycentric mapping $\phi$ of $N(\mathfrak{U})$ onto $G(\mathfrak{U})$ which maps $u_\alpha$ onto $a_\alpha$ is continuous. Let $g = \phi f$. Then, since $\phi^{-1}\operatorname{St} a_\alpha = \operatorname{St} u_\alpha$,

$$g^{-1}\operatorname{St} a_\alpha = f^{-1}\phi^{-1}\operatorname{St} a_\alpha = f^{-1}\operatorname{St} u_\alpha \subset U_\alpha.$$

Hence $g$ is a canonical mapping of $R$ into $G(\mathfrak{U})$.

Conversely, let $g$ be any canonical mapping of $R$ into $G(\mathfrak{U})$. The natural nerve of the covering $\{\operatorname{St} a_\alpha\}$ of $G(\mathfrak{U})$ by the stars of its vertices may be identified with $N(\mathfrak{U})$. It can be shown that $G(\mathfrak{U})$ is a paracompact normal (metrizable) space [6, Theorem 2]. Hence, by (c), there is a mapping $\psi$ of $G(\mathfrak{U})$ into $N(\mathfrak{U})$ canonical with respect to the covering $\{\operatorname{St} a_\alpha\}$. Thus $\psi^{-1}\operatorname{St} u_\alpha \subset \operatorname{St} a_\alpha$. Let $f = \psi g$. Then

$$f^{-1}\operatorname{St} u_\alpha = g^{-1}\psi^{-1}\operatorname{St} u_\alpha \subset g^{-1}\operatorname{St} a_\alpha \subset U_\alpha.$$

Hence $f$ is a canonical mapping of $R$ into $N(\mathfrak{U})$.

Applying Theorem 4 to Theorems 1 and 2, we have:

COROLLARY 1. *There exists a canonical mapping of a normal space $R$ into the geometric nerve of a covering $\mathfrak{U}$ if and only if $\mathfrak{U}$ has a locally finite refinement.*

COROLLARY 2. *There exists a canonical mapping of a space $R$ into the geometric nerve of an arbitrary covering if and only if $R$ is paracompact and normal.*

It likewise follows from Theorem 4 that the theorems of Lefschetz on canonical mappings into geometric nerves hold also for mappings into natural nerves. In particular, using his notion of "analytic" covering and his fundamental mapping theorem [10, p. 41], we have:

COROLLARY 3. *There exists a canonical mapping of a space $R$ into the natural nerve of a point-finite covering $\mathfrak{U}$ if and only if $\mathfrak{U}$ is analytic.*

## BIBLIOGRAPHY


1. P. Alexandroff, *Sur la dimension des ensembles fermés*, C. R. Acad. Sci. Paris vol. 183 (1926) pp. 640–642.

2. ———, *Über den allgemeinen Dimensionsbegriff und seine Beziehungen zur elementaren geometrischen Anschauung*, Math. Ann. vol. 98 (1928) pp. 617–636.

3. ———, *Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension*, Ann. of Math. (2) vol. 30 (1928) pp. 101–187.

4. J. Dieudonné, *Une généralisation des espaces compacts*, J. Math. Pures Appl. (9) vol. 23 (1944) pp. 65–76.

5. C. H. Dowker, *Mapping theorems for non-compact spaces*, Amer. J. Math. vol. 69 (1947) pp. 200–242.

6. ———, *An imbedding theorem for paracompact metric spaces*, Duke Math. J. vol. 14 (1947).

7. W. Hurewicz, *Über Abbildungen von endlichdimensionalen Räumen auf Teilmengen Cartesischer Räume*, Preuss. Akad. Wiss. Sitzungsber. (1933) pp. 754–768.

8. C. Kuratowski, *Sur un théorème fondamental concernant le nerf d'un système d'ensembles*, Fund. Math. vol. 20 (1933) pp. 191–196.

9. ———, *Sur le prolongement des fonctions continues et les transformations en polytopes*, Fund. Math. vol. 24 (1935) pp. 259–268.

10. S. Lefschetz, *Topics in topology*, Annals of Mathematics Studies, no. 10, Princeton, 1942.

11. ———, *On the mapping of abstract spaces on polytopes*, Proc. Nat. Acad. Sci. U.S.A. vol. 25 (1939) pp. 49–50.

12. J. W. Tukey, *The intrinsic metric of a polytope*, Proc. Nat. Acad. Sci. U.S.A. vol. 25 (1939) p. 51.


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