BOOK REVIEWS

Eigenfunction expansions associated with second order differential equations. By E. C. Titchmarsh. Oxford, Clarendon Press, 1946. 8+174 pp. $7.00.

The subject of this book has its origin in the Sturm-Liouville expansions. The author deals with the problem of expanding an arbitrary function in terms of the e.f's (eigenfunctions) of a second order ordinary differential equation, with emphasis on the singular theory. He puts aside H. Weyl's method of handling of the singular theory on the basis of integral equations and also bypasses the use of the general theory of linear operators in Hilbert spaces (M. H. Stone); instead, use is made of contour integrations and the Cauchy calculus of residues. The material in Chapters 1, 2, 4, 5, 7 is in some essential parts due to the author; Chapters 9, 10 are entirely due to him; the highly important Chapter 3 involves some very heavy technical equipment and is due essentially to the author and to H. Weyl; the material in Chapter 6 is due to M. H. Stone. The book is written with complete rigor in a very readable style. Specifically is studied the operator $L \equiv q(x) - d^2/dx^2$, where $q(x)$ is a given function, defined on some finite or infinite interval (a, b). The values of λ and the corresponding solutions (subject to suitable boundary conditions) of

$$(1) \qquad Ly = \lambda y$$

are termed the e.v's (eigenvalues) and e.f's, respectively.

In the regular case (chap. 1) the following is proved. If real q is continuous on a finite interval (a, b), then (1) has a solution ϕ so that $\phi(a) = \sin \alpha$, $\phi'(a) = -\cos \alpha$ (α assigned); if, in addition, f is integrable over (a, b), then the Sturm-Liouville expansion $s(x)$ behaves with regard to convergence as an ordinary Fourier series, while $2^{-1}(f(x+0)+f(x-0)) = s(x)$ when f is of bounded variation near x. In chap. 2 the author treats singular cases when the expansion is still a series and the interval is $(0, \infty)$. It is assumed that q is continuous on every finite subinterval of $(0, \infty)$. The general solution of (1) is of the form $\theta + l\phi$, where θ, ϕ are solutions of (1) such that $\theta(0) = -\phi'(0) = \cos \alpha$, $\theta'(0) = \phi(0) = \sin \alpha$. If solutions are considered for which $\{\theta(b) + l\phi(b)\} \cos \beta + \{\theta'(b) + l\phi'(b)\} \sin \beta = 0$, one obtains $l = l(\lambda) = -[\theta(b) \operatorname{ctg} \beta + \theta'(b)][\phi(b) \operatorname{ctg} \beta + \phi'(b)]^{-1}$. As β varies, l describes a circle C_b; $C_b \to$ a "limit-circle" or a "limit-point" as $b \to \infty$. For every non-real λ, (1) has a solution $\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\phi(x, \lambda) \subset L^2(0, \infty)$, where $m(\lambda)$ is the limit-point or is any point on the limit-circle. It is assumed that the only singularities of $m(\lambda)$ are poles $\lambda_0, \lambda_1, \cdots$, with corresponding residues $n_0, n_1, \cdots$. If f and Lf are $L^2(0, \infty)$, $f(0) \cos \alpha + f'(0) \sin \alpha = 0$ and the $\lim_{x\to\infty} W(\psi(x, \lambda), f(x)) = 0$ (W is the wronskian) for all non-real λ, then

$$f(x) = \sum c_n\psi_n(x)). \tag{2}$$

Moreover $\int_0^\infty f^2dx = \sum c_n^2$. Form $\Phi(x, \lambda) = \psi(x, \lambda)\int_0^\infty \phi(y, \lambda)f(y)dy + \phi(x, \lambda)\int_x^\infty \psi(y, \lambda)f(y)dy$; if f is $L^2(0, \infty)$ and λ is distinct from the λ_n, then $\Phi(x, \lambda) = \sum c_n(\lambda - \lambda_n)^{-1}\psi_n(x)$. Indications are given for the case $(-\infty, \infty)$.

In the general singular case (chap. 3) $m(\lambda)$ is not restricted. Form $k(\lambda) = \lim_{\delta\to 0} \int_0^\lambda [-\operatorname{Im}(u+i\delta)]du$ $(\delta > 0)$. If f is $L^2(0, \infty)$, then the $g_n(\lambda) = \int_0^n \phi(y, \lambda)f(y)dy$ converge in the mean square with respect to $k(\lambda)$ over $(-\infty, \infty)$ to a limit $g(\lambda)$, while $\pi\int_0^\infty f^2dx = \int_{-\infty}^\infty g^2(\lambda)dk(\lambda)$. The author gives some indications for the case $(-\infty, \infty)$. The concept of s. (spectrum) enters as follows. When $(a, b) = (0, \infty)$ the s. is the λ-set which is the complement of the set of points in the neighborhood of which $k(\lambda)$ is constant. It is only the s. that contributes in the representation formula $\pi f(x) = \int_{-\infty}^\infty \phi(x, u)d\int_0^\infty Q(y, u)f(y)dy$, where $Q(y, u) = \int_0^u \phi(y, \lambda)dk(\lambda)$ (valid when f is subject to the conditions imposed in (2)). If $m(x)$ is meromorphic, there is a point-s. (consisting of the set of poles). The continuous s. is the set of intervals in each of which $k(\lambda)$ is steadily increasing.

Chap. 4 contains numerous applications to Hermite, Legendre, Bessel, and so on, expansions. In chap. 5 the following is proved. If q is $L(0, \infty)$, then the s. is continuous on $(0, \infty)$ and there is a point-s. on $(-\infty, 0)$; when $q \to \infty$ (steadily), $q' \geqq 0$, $q' = O(|q|^c)$ $(0 < c < 3/2)$ and q'' is eventually of constant sign, then the s. is discrete. Let $q \leqq 0$, $q' < 0$, $q \to -\infty$, $q' = O\{|q|^c\}$ $(0 < c < 3/2)$ and q'' be of constant sign; then, if $\sigma = \int^\infty |q|^{-1/2}dx = \infty$, there is a continuous s. over $(-\infty, \infty)$ and, if $\sigma < \infty$, a point s. on $(0, \infty)$ and a continuous s. on $(-\infty, 0)$. When $q \to \infty$ as $x \to \infty$, there are discrete e.v's, while the e.f associated with λ_n has n zeros.

In chap. 6 it is shown that the expansion of f, given in chap 3, will hold when the conditions on f are relaxed, while additional ones are imposed on q. The expansion of f is given when q is $L(0, \infty)$ and f is $L^2(0, \infty)$ and also $L(0, \infty)$. In chap. 7 a study is presented of the discrete e.v's $\lambda_0, \lambda_1, \cdots$ when $q \to \infty$. The following is one of the results. Let $N(\lambda) = n$ $(\lambda_{n-1} < \lambda \leqq \lambda_n)$; if $q' \to \infty$, $q'' \geqq 0$, $q'' \leqq (q')^\gamma (1 < \gamma < 4/3;\ x > x_0)$, then $\pi N(\lambda) = \int_0^p (\lambda - q(x))^{1/2}dx + O(1)$ (here $p = p(\lambda)$ arises from the equation $q(p) = \lambda$). In chap. 8 a further approximation to $N(\lambda)$ is given on the basis of a rigorous method, suggested by a heuristic argument used by physicists in connection with the quantum mechanics equation $\psi''+8\pi^2mh^{-2}(E-V(x)\psi=0$ (Brillouin, Wentzel, Kramer). In chap. 9 it is proved that if $q'>0$, $q''\geqq 0$, $q''\leqq(q')^\gamma(1<\gamma<4/3;\ x\geqq x_0)$ and f is $L^2(0,\ \infty)$, then $f(x+0)+f(x-0)=2\sum c_n\psi_n(x)$, provided f is of bounded variation near x. In chap. 10 the author proves the following summability theorem. If continuous $q\rightarrow\infty$ monotonically and f is $L^2(0,\ \infty)$, then $f(x)=\lim_{v\rightarrow\infty}\sum_n(v+\lambda_n)^{-1}vc_n\psi_n(x)$ for every x for which $\int_0^\eta|f(x+y)-f(x)|\,dy=o(\eta)$, as $\eta\rightarrow 0$ (that is, almost everywhere). With the aid of the above result the expansion theorem of chap. 9 is then proved anew, which presents an analogy with the situation in the ordinary Fourier theory.

Some developments analogous to those in the book under review have been carried out in the field of partial differential equations by T. Carleman [Arkiv för Mathematik, Astronomi och Fysik vol. 24 B (1934) pp. 1–7] and by the present reviewer [Ann. of Math. vol. 43 (1942) pp. 1–55; also, Rec. Math. (Mat. Sbornik) N.S. 20 (1947) pp. 365–430]. The field of partial differential equations being so much more difficult than that of ordinary differential equations, much more remains to be done. The book of Titchmarsh may serve as a useful guide in the line of investigation just mentioned.

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The theory of functions of real variables. By L. M. Graves. New York and London, McGraw-Hill, 1946. 10+300 pp. $4.00.

The theory of functions of real variables occupies a central position in present day analysis, and it is typical for graduate schools in the United States to offer students in mathematics a one year introduction to the subject. It has always been a problem to find a suitable text for such a course, needing, as it does, something shorter and crisper than one of the standard treatises. The volume under review is written with this end in view.

The first chapter is a short exposition of the ideas and methods of deductive logic. The notions of negation, conjunction and alternation of propositions are given symbolic notations, and regarded as undefined (but "generally understood") operations: a list of laws by which their use is to be governed (for example, double negation, excluded middle) is discussed. The calculus of classes is briefly described, and the student is introduced to a technique of translating English sentences involving logical and class relationships into briefer symbolic formulae. We are warned about possible paradoxes which may arise through unguarded use of these notions, and the writer de-