ON THE FACTORISATION OF ORTHOGONAL TRANSFORMATIONS INTO SYMMETRIES

H. C. LEE

Consider the quadratic form

\[ g_{11}(x^1)^2 + g_{22}(x^2)^2 + \cdots + g_{nn}(x^n)^2 \]

where each \( g_{ii} \) is +1 or -1 (but fixed once for all). A homogeneous linear transformation in the variables \( x^1, \ldots, x^n \) (or its matrix) is called orthogonal if it leaves this form invariant. If and only if all \( g_{ii} \) have the same sign we have orthogonality in the classical sense. In any sense it can be easily seen that the orthogonal transformations form a group and that they have determinants \( \pm 1 \).

All quantities considered may be either in the real field or in the complex field (or in any field whatever).

Let \( g_{ij} = 0 \) (\( i \neq j \)). Then, if \( s^i \) is a contravariant vector, its covariant is defined by \( s_i = g_{ik}s^k \) (summation convention). The vector \( s \) is said to be isotropic if it has zero length, that is, if \( s^hs_h = 0 \). If \( s \) is not an isotropic vector, the matrix

\[
\delta^i_k - 2 s^i s_k/s^h s_h
\]

where \( \delta^i_k = 1 \) or 0 if \( i = k \) or \( i \neq k \), defines a special orthogonal transformation of determinant \( -1 \) which we call a symmetry.\(^1\)

Now, by a series of arguments, E. Cartan has proved the following theorem.

**Theorem.**\(^2\) Every orthogonal transformation is decomposable into the product of a number not greater than \( n \) of symmetries.

I shall give a short proof\(^3\) of this theorem below:

Let \( a^i_k \) be the matrix of an arbitrarily given orthogonal transformation. Form the product matrix

\[
b^i_k = s^i_j a^j_k = a^i_k - 2 s^i s^j a^j_k / s^h s_h
\]

Received by the editors July 22, 1947.

\(^1\) E. Cartan, *La théorie des spineurs*, vol. 1, p. 13.


\(^3\) The first half of the present proof is similar to a part of another simple proof given by W. Givens, *Factorisation and signatures of Lorentz matrices*, Bull. Amer. Math. Soc. vol. 46 (1940) p. 82, but Givens' proof only establishes the weaker statement that the number of symmetries in the factorisation is not greater than \( 2n \).
and consider its first column $b_1^i$. We show that one can choose the vector $s$ such that $b_1^i = \delta_1^i$. In fact, it suffices for this to take $s^i = \rho (a_1^i - \delta_1^i)$, $\rho \neq 0$. Then we find

$$s^h s_h = g_{hh} s^h s = 2\rho^2 (g_{11} - g_{11} a_1^i) = 2\rho^2 g_{11} (1 - a_1^i).$$

(1) If $a_1^i \neq 1$, we have $s^h s_h \neq 0$ and so $s$ is not an isotropic vector. Then

$$b_1^i = a_1^i - 2s^i a_1^i / 2\rho^2 g_{11} (1 - a_1^i) = \delta_1^i.$$

This being the first column of the matrix $b_1^i$, the orthogonality between this column and each of the other columns implies that the matrix $b$ has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$$

where $c$ is evidently an orthogonal matrix of order $n - 1$. Hence, by induction, the theorem is proved in this case.

(2) If $a_1^i = 1$, we can evidently transform the given orthogonal transformation by another orthogonal transformation so as to render $a_1^i \neq 1$, and we have only to observe that the symmetries are transformed again into symmetries by an orthogonal transformation.

CAMBRIDGE, ENGLAND