ON THE LOCATION OF THE ZEROS OF THE DERIVATIVES OF A POLYNOMIAL SYMMETRIC IN THE ORIGIN

J. L. WALSH

If the zeros of a polynomial $p(z)$ when plotted in the z-plane are symmetric in 0: $z=0$, the zeros of the derivative $p'(z)$ of $p(z)$ can profitably be studied by transforming onto the w-plane, with $w=z^2$, and applying known theorems there.[1] It is the purpose of the present note to carry that study somewhat farther than has been previously done, in particular to consider the higher derivatives of $p(z)$.

Under the transformation $w=u+iv=z^2=(x+iy)^2$, an arbitrary line $Au+Bv+C=0$ in the w-plane corresponds to an equilateral hyperbola $A(x^2-y^2)+2Bxy+C=0$ in the z-plane with center 0 or to two perpendicular lines intersecting at 0. A half-plane in the w-plane for which $w=0$ is an interior or exterior point corresponds in the z-plane respectively to the exterior or interior of an equilateral hyperbola whose center is 0; a half-plane for which $w=0$ is a boundary point corresponds to a double sector with vertex $z=0$ and angle $\pi/2$. A point z is considered to be exterior or interior to a hyperbola according as the curve at its nearest point is convex or concave toward z.

We write the given polynomial in the form

$$p(z) = z^l \prod_{j=1}^{q} (z^2 - \alpha_j^2), \qquad \alpha_j \neq 0, \tag{1}$$

and in the w-plane study the polynomials ($w=z^2$)

$$P(w) = P(z^2) = [p(z)]^2, \qquad P'(w) = p(z)\cdot p'(z)/z. \tag{2}$$

Each zero of $P(w)$ corresponds to a zero of $p(z)$ and reciprocally; each zero of $P'(w)$ corresponds to a zero of $p(z)$ or $p'(z)$ and reciprocally except that $z=0$ is a zero of $p'(z)$ unless $z=0$ is a simple zero of $p(z)$.

We have (loc. cit.) by Lucas' Theorem

THEOREM 1. *If the zeros of $p(z)$ are symmetric in 0 and lie in the closed exterior of an equilateral hyperbola with center 0 or in the closed exterior of a double sector with vertex 0 and angle $\pi/2$, then the zeros of $p'(z)$ lie also in that closed exterior.*

If the zeros of $p(z)$ are symmetric in 0 and lie in the closed interior of an equilateral hyperbola with center 0, then the zeros of $p'(z)$ also lie in that closed interior except for a simple zero at 0.

Received by the editors December 4, 1947.

[1] Walsh, Mathematica vol. 8 (1933) pp. 185–190.

To the polynomial $P(w)$ we now apply the theorem:[2]

Let $P(w)$ be a polynomial in w of degree n with an l-fold $(l>0)$ zero at $w=0$ and all the remaining zeros of $P(w)$ in the closed half-plane Π *not containing $w=0$. Then except for a zero at $w=0$ of multiplicity $l-1$, all zeros of $P'(w)$ lie in the closed half-plane obtained by shrinking* Π *toward the origin in the ratio $n:l$.*

The degrees of $p(z)$ and $P(w)$ defined by (1) and (2) are equal, as are the multiplicities of their zeros $z=0$ and $w=0$. Under the transformation $w=z^2$, the line $u=a$ corresponds to the hyperbola $x^2-y^2=a$, and the line $u=la/n$ corresponds to $x^2-y^2=la/n$, so we have

THEOREM 2. *Let $p(z)$ be a polynomial of degree n whose zeros are symmetric in the origin* 0, *let* 0 *be an l-fold zero $(l>0)$, and let all the other zeros lie in the closed interior of an equilateral hyperbola H with center* 0. *Then except for an $(l-1)$-fold zero at* 0, *all zeros of $p'(z)$ lie in the closed interior of the hyperbola obtained by shrinking H toward* 0 *in the ratio* $(n:l)^{1/2}$.

We turn now to the higher derivatives of the polynomial $p(z)$. Under the conditions of the first part of Theorem 1, the zeros of every derivative of $p(z)$ not vanishing identically lie in the given closed region.

Suppose, however, the zeros of the polynomial $p(z)$ of degree n lie in the closed interior of an equilateral hyperbola H whose center is 0. The polynomial $p'(z)$ has a simple zero at 0, and by Theorem 1 the remaining zeros of $p'(z)$ lie in the closed interior of H. By Theorem 2, all zeros of $p''(z)$ lie in the closed interior of the hyperbola obtained by shrinking H toward 0 in the ratio $[(n-1):1]^{1/2}$. The higher derivatives of $p(z)$ not vanishing identically have alternately a simple zero at 0 and no zero at 0. Continued application of Theorems 1 and 2, in alternation, then yields further hyperbolas which respectively contain the zeros other than 0 of the kth derivative of $p(z)$. However, except in the cases $k=1$, and $k=2$ with $n\geqq 4$, these hyperbolas are not the most favorable that can be obtained; a zero of $p'(z)$ in Theorem 2 cannot lie on the new hyperbola unless all zeros of $p(z)$ lie on H in two points symmetric in 0. We proceed to prove the following generalization of Theorem 2, the principal result of the present note:

THEOREM 3. *Let $p(z)$ be a polynomial of degree n whose zeros are symmetric in* 0, *let* 0 *be an l-fold zero, and let the remaining zeros lie in the closed interior of a hyperbola H whose center is* 0. *Let $p_0(z)$ be the polynomial of degree n whose zeros are symmetric in* 0, *which has an l-fold*

---

[2] Walsh, Trans. Amer. Math. Soc. vol. 22 (1921) pp. 101–116; p. 115.

zero at 0, *and whose remaining zeros lie at the vertices of* H. *Let* z_k, $1 \leqq k \leqq n-2$, *be one of the zeros of* $p_0^{(k)}(z)$ *of smallest positive modulus, and let* H_k *be the equilateral hyperbola with center* 0 *which has* z_k *as a vertex. Then all zeros of* $p^{(k)}(z)$ *other than* 0 *lie in the closed interior of* H_k.

Of course 0 need not be a zero of $p^{(k)}(z)$; whether or not 0 is such a zero depends on l, n, and k. To prove Theorem 3 we need a more powerful result than that used in proving Theorem 2:

Let $Q(w)$ *be a polynomial in* w *of degree* q, *and let the numbers* A_j *be constants. If the locus of the zeros of* $Q(w)$ *is a closed half-plane* Π, *then the locus of the zeros of the polynomial*

$$A_0 w^r Q(w) + A_1 w^{r+1} Q'(w) + \cdots + A_q w^{r+q} Q^{(q)}(w) \tag{3}$$

is also a number of half-planes, identical with the locus of the zeros of the polynomial (3) *with* $Q(w)$ *replaced by* $(w-\alpha)^q$:

$$\begin{aligned} &A_0 w^r (w-\alpha)^q + qA_1 w^{r+1}(w-\alpha)^{q-1} + \cdots \\ &\qquad\qquad + q(q-1)\cdot \cdots \cdot 1 A_q w^{r+q} \end{aligned} \tag{4}$$

when the locus of α *is* Π.

The theorem just quoted is essentially a special case of a more general theorem[3] concerning a linear combination of products of derivatives of two polynomials. If polynomial (4) is written as the product of w^{r+q} and a polynomial in $W=(w-\alpha)/w$ whose zeros are $W=W_1, W_2, \cdots, W_q$, it is seen that the common locus of the zeros of (3) and (4) consists of the origin $w=0$ (provided $w=0$ is a zero of (4)) plus a number of half-planes, loci of the points $w=\alpha/(1-W_j)$, $j=1, 2, \cdots, q$, when the locus of α is Π; if Π does not contain the point $w=0$, a zero $W_j=1$ is to be ignored; if Π contains the point $w=0$ and if $W_j=1$ is a zero of (4), the entire plane is to be considered the locus of the zeros of (3) and (4). If Π does not contain 0, and if for real positive α all zeros of (4) lie in the interval $0 \leqq w \leqq \alpha$, then the locus of the zeros of (4) is the origin (if $w=0$ is a zero of (4)) plus a half-plane not containing 0 bounded by a line parallel to the boundary of Π, traced by the zero of (4) nearest to but different from 0 when α traces the boundary of Π.

In Theorem 3 we write

$$\begin{aligned} p(z) &= z^l p_1(z^2), \qquad\qquad p_1(0) \neq 0, \\ p'(z) &= l z^{l-1} p_1(z^2) + 2z^{l+1} p_1'(z^2), \end{aligned}$$

[3] Walsh, Trans. Amer. Math. Soc. vol. 24 (1922) pp. 163–180; Theorem 9 of that paper applies to two circular regions, here specialized to a single point and a half-plane respectively.

$$p''(z) = l(l-1)z^{l-2}p_1(z^2) + (4l+2)z^l p_1'(z^2) + 4z^{l+2}p_1''(z^2),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .$$

Except perhaps for a factor z, these derivatives are precisely of the form (3), with $w=z^2$. If we have $p_0(z)=z^l(z^2-\beta^2)^{(n-l)/2}$, the zeros of $p_0^{(k)}(z)$ lie on the line segment joining β and $-\beta$, and the corresponding zeros in the w-plane lie on the segment joining 0 and β^2; the images in the w-plane of the zeros of $p_1(z)$ lie in a half-plane Π not containing $w=0$; thus by the theorem quoted the images in the w-plane of the zeros of $p^{(k)}(z)$ other than $z=0$ lie in the half-plane whose boundary is parallel to that of Π, which contains Π in its interior, and whose boundary passes through the image of the zero of $p_0^{(k)}(z)$ nearest to but distinct from 0. Theorem 3 follows.

When as in Theorem 1 we study the zeros of a polynomial $p(z)$ that are symmetric in 0, equilateral hyperbolas with center 0 are both (1) the lines of force in the z-plane in the Gaussian field due to two particles symmetric in 0 and (2) the images in the z-plane of the straight lines in the w-plane under the transformation $w=z^2$. Indeed, the lines of force due to particles at $z=\alpha$ and $z=-\alpha$ are loci arg $(z^2-\alpha^2)=$const., arcs of equilateral hyperbolas with center 0; half-lines in the w-plane can be written arg $(w-w_0)=$const., and their images in the z-plane are loci arg $(z^2-w_0)=$const.

A set of points in the z-plane is said to have *m-fold symmetry about* 0 if the set is unchanged by a rotation about 0 through an angle of $2\pi/m$. A polynomial $p(z)$ whose zeros possess m-fold symmetry about 0 is readily studied by means of the transformation $w=z^m$. Under this transformation a straight line in the w-plane not through $w=0$ corresponds to a curve which may be called an *m-hyperbola* with center 0. A closed half-plane in the w-plane containing 0 in its interior corresponds to the closed exterior of an m-hyperbola, and a closed half-plane not containing 0 corresponds to the closed interior of an m-hyperbola. A line through $w=0$ corresponds to m equally spaced lines through $z=0$, and a half-plane in the w-plane bounded by a line through $w=0$ corresponds to m equally spaced sectors in the z-plane, each of angle π/m. The function $[p(w^{1/m})]^m$ is a polynomial in w which is readily studied in the w-plane, and yields results on the zeros of $p'(z)$ in the z-plane. Theorem 3 extends directly to a polynomial $p(z)$ whose zeros possess m-fold symmetry in 0. The m-hyperbolas with center 0 are not merely the images in the z-plane of the straight lines in the w-plane, but also the lines of force in the z-plane due to m particles m-fold symmetric in $z=0$.

HARVARD UNIVERSITY