ON POLYNOMIALS AND LAGRANGE’S FORM
OF THE GENERAL MEAN-VALUE THEOREM

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Suppose that in \((a<x<b)\) (hereafter referred to as \((a, b)\)),

(1) \(f(x)\) is defined and has derivatives of the first \(n\) orders.

Then, from the general mean-value theorem with Lagrange’s form of
remainder follows the existence of \(\theta(\xi, h)\), such that

\[
\begin{align*}
f(x + h) &= f(x) + \sum_{r=1}^{n-1} \frac{h^r}{r!} f^{(r)}(x) + \frac{h^n}{n!} f^{(n)}(x + \theta h) \\
& \quad \text{for } a < x < x + h < b.
\end{align*}
\]

The \(\theta\) in (2) is sometimes a uniquely determinate function of \(x\) and \(h\)
in the relevant domain \(a < x < x + h < b\) (hereafter referred to as \(R\)),
as, for instance, if \(f^{(n+1)}(x)\) exists and is not zero in \((a, b)\). If, further,
\(f^{(n+1)}(x)\) is continuous in \((a, b)\), it is easily seen that

\[
\lim_{h \to 0} \theta(x, h) = \frac{1}{n+1} \quad \text{in } a < x < b.
\]

It is also possible for \(\theta(x, h)\) to be an analytic function, for example,

\[
\theta(x, h) = h^{-1} \log \left( 1 + \sum_{r=1}^{\infty} \frac{h^r T(n + 1)}{T(n + r + 1)} \right),
\]

which happens when \(f(x) = e^x\).

It would, therefore, seem worth while to determine the types of
functions that are or are not possible for \(\theta(x, h)\). Inquiry in this
direction has led to the results of this paper, namely:

**Theorem 1.** If a polynomial \(\theta(x, h)\) exists such that (2) is true with
\(\theta(x, h)\) in place of \(\theta\), then \(f^{(n+1)}(x)\) exists in \((a, b)\) and either

(a) \(f^{(n+1)}(x) = 0\) in \((a, b)\)

or

(b) \(f^{(n+1)}(x) = \text{a constant } \neq 0 \text{ in } (a, b), \text{ and } \theta(x, h) \text{ is uniquely determinate}
\text{ and equal to } 1/(n+1) \text{ in } R\).

**Theorem 2.** If (2) is true with \(\theta(x, h) = c(x) + h^r \phi(x, h)\) where

(3) \(\phi(x, h)\) is bounded in \(R\);

(4) \(d\) is a constant greater than 1;

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(5) $\frac{\partial \theta}{\partial x}, \frac{\partial^2 \theta}{\partial^2 x}$ are continuous in $x$, and $\theta$ is bounded in $R$;
(6) for all sufficiently small $h$, $1 + h(\frac{\partial \theta}{\partial x}) \neq 0$ in $R$;
then, also, (a) and (b) of Theorem 1 are true.

It is significant that, if $\theta$ is uniquely determined by (2) in $R$ and not equal to $1/(n+1)$, then $\theta = \theta(x, h)$ cannot be equal to a polynomial in $R$ (by Theorem 1) or even to an analytic function (by Theorem 2) satisfying

(7) $\lim_{h \to 0} \frac{\partial \theta}{\partial h} = 0$ for every $x$ in $(a, b)$.

In the following we write $\theta(x, 0)$ for $\lim_{h \to 0} \theta(x, h)$ and $\theta_1(x, 0)$ for $\lim_{h \to 0} (\theta(x, h) - \theta(x, 0))/h$ (which limits obviously exist in the contexts of the two theorems), and $\theta_{n+2}$ for $(\partial^{n+2}/\partial x^n h^n)\theta$, wherever the latter obviously exists.]

Proof of Theorem 1.

(8) The conditions (5) and (6) are obviously satisfied here and (2) is true by hypothesis.

On account of the consequent boundedness of $\theta$ in $R$, and the continuity of $\theta$ in $x$, follows

(9) $y = x + \theta h$ for every $y$ in $(a, b)$, with any sufficiently small $h$ and a corresponding $x$ such that $(x, h)$ lies in $R$.

From (8) and (9) follows

(10) $f^{n+1}(x)$ and $f^{n+2}(x)$ exist and are continuous in $(a, b)$.

Now, from the general mean-value theorem follows

$$f(x + h) = f(x) + \sum_{r=1}^{n} \frac{h^r}{r!} f^{(r)}(x) + \frac{h^{n+1}}{(n + 1)!} f^{(n+1)}(x)$$

$$+ \frac{h^{n+2}}{(n + 2)!} f^{(n+2)}(x + \theta_1 h), \quad 0 < \theta_1 < 1, \quad (x, h) \subset R;$$

and from (2) and the same theorem applied to $f^n(x + \theta h)$ follows

$$f(x + h) = f(x) + \sum_{r=1}^{n} \frac{h^r}{r!} f^{(r)}(x) + \frac{h^{n+1}}{(n)!} f^{(n+1)}(x)$$

$$+ \frac{h^{n+2} \theta^2}{n!} f^{(n+2)}(x + \theta_2 h), \quad 0 < \theta_2 < 1, \quad (x, h) \subset R.$$

Subtracting (12) from (11) and making $h \to 0$ after division by $h^{n+1}$, it follows by (10) that

$$f^{(n+1)}(x) \cdot [1 - (n + 1)\theta(x, 0)] = 0.$$

Using (13) in (11) and (12), and making $h \to 0$ after division of their difference by $h^{n+2}$, it follows, again by (10),
\[
\begin{align*}
\text{(14)} & \quad f^{(n+2)}(x) \left[ 1 - \frac{(n+1)(n+2)}{2} \theta^2(x, 0) \right] \\
& \quad \quad \quad \quad \quad \quad \quad \quad - f^{(n+1)}(x)(n+1)(n+2)\theta_1(x, 0) = 0.
\end{align*}
\]

Now, either

\begin{align*}
\text{(15a)} & \quad f^{(n+1)}(x) = 0 \text{ everywhere in } (a, b); \\
\text{(15b)} & \quad f^{(n+1)}(x) \neq 0 \text{ everywhere in } (a, b); \\
\text{(15c)} & \quad \text{on account of the continuity (by (10)) of } f^{(n+1)}(x) \text{ there exists a closed interval } (a_1, b_1) \text{ contained in } (a, b) \text{ such that } f^{(n+1)}(x) \neq 0 \text{ for } a_1 < x < b_1, \text{ and one at least of } f^{(n+1)}(a_1) \text{ and } f^{(n+1)}(b_1) \text{ is zero.}
\end{align*}

If (15c) were possible, then we should have, by (13) and (14),
\[
f^{(n+2)}(x) \cdot n/2(n+1) - f^{(n+1)}(x)(n+1)(n+2)\theta_1(x, 0) = 0
\]
in \((a_1 < x < b_1),
\]
and hence \(f^{(n+1)}(x) = A \cdot \exp \{ \phi(x) \} \) in \(a_1 < x < b_1\), where \(\phi(x)\) is a polynomial and \(A\) is a constant, and making \(x \to a_1 \) or \(b_1\) in this, there would follow that \(f^{(n+1)}(x) = 0\) in \(a_1 < x < b_1\), which contradicts (15c). Hence (16) (15c) is impossible, and \(f^{(n+1)}(x) = A \exp \{ \phi(x) \} \) in \(a < x < b\), where \(\phi(x)\) is a polynomial and \(A = a\) constant \(\neq 0\), if \(f^{(n+1)}(x) \neq 0\) for some \(x\) in \((a, b)\).

Now differentiating (2) with respect to \(x\) and \(h\), as is obviously permissible on account of (10), and subtracting, and dividing by \(h^{n-1}\), it follows that
\[
f^{(n)}(x) - f^{(n)}(x + \theta h) = \frac{h}{n} f^{(n+1)}(x + \theta h)[\theta - 1 + h\theta_{01} - h\theta_{10}] \text{ in } R.
\]

Differentiating this (possible by (10)) with respect to \(x\) and using (16) we get
\[
\text{(17)} \quad \exp \{ k(x, h) \} = g(x, h) \text{ in } R, \text{ in case } (15b), \text{ where } k(x, h) = \phi(x) - \phi(x + \theta h) \text{ and } k(x, h) \text{ and } g(x, h) \text{ are polynomials in } x \text{ and } h.
\]

It is now seen by the theory of analytic continuation that (17) is impossible unless \(k(x, h)\) is a constant, which again is seen to be zero by keeping \(x\) fixed and making \(h \to +0\). Hence
\[
\text{(18)} \quad \phi(x) = \phi(x + \theta h) \quad \text{in } R.
\]

Now from (2) obviously follows
\[
\text{(19)} \quad f(x) \text{ is a polynomial of degree not greater than } n \text{ in } (a, b) \text{ if } \theta(x, h) = 0. \text{ Also, by continuous variation of } x \text{ and } h \text{ in } R \text{ it follows from } (18) \text{ that}
\]
\( \phi(x) = k \) is a constant \( k \) in \( (a, b) \) if \( \theta(x, h) \neq 0 \), and hence, using (16), follows

(21) \( f^{(n+1)}(x) = Ae^k \) in \( (a, b) \) where \( A \neq 0 \) if \( f^{(n+1)}(x) \neq 0 \) in \( (a, b) \).

Now the theorem follows from (10), (15a), (15b), (16), (19) and (21).

since, when \( f^{(n+1)}(x) = k \) is a constant \( k \neq 0 \), \( \theta = 1/(n+1) \) and is uniquely determined by (2) in \( R \).

**proof of theorem 2.** In this case, the statements (8) to (14) follow as above, and \( \theta_1(x, 0) = 0 \) since \( d > 1 \). Hence (13) and (14) now become

(22) \( f^{(n+1)}(x) \left[ 1 - (n + 1)c(x) \right] = 0 \) in \( (a, b) \),

(23) \( f^{(n+2)}(x) \left[ 1 - \frac{(n + 1)(n + 2)}{2} c^2(x) \right] = 0 \) in \( (a, b) \).

Hence either

(24a) \( f^{(n+1)}(x) = 0 \) everywhere in \( (a, b) \),

or

(24b) \( f^{(n+1)}(x) = c \neq 0 \) for some \( x \) in \( (a, b) \).

Then, (22) and (23) give

(25) \( c(x) = 1/(n + 1) \) wherever \( f^{(n+1)}(x) \neq 0 \),

(26) \( f^{(n+2)}(x) = 0 \) wherever \( f^{(n+1)}(x) \neq 0 \).

The theorem now follows from (10), (24a), (24b), (25) and (26), since, when \( f^{(n+1)}(x) = c \neq 0 \) in \( (a, b) \), \( \theta \) in (2) is uniquely determined in \( R \).

Note added January 18, 1948. The conclusions (a) and (b) of Theorem 1 are true if (2) holds with \( \theta(x, h) \) in place of \( \theta \), where

\[ \theta(x, h) = \sum_{r=0}^{m} h^r \theta_r(x), \]

and \( \theta_r(x) \) is a polynomial, \( \theta(x, h) \) satisfies (6) and each of the functions \( \theta_r(x) \) satisfies (5). The line of proof is briefly as follows:

The arguments up to and including (16) are the same as above, and the equation in (17) is now true with \( K(x, h) = \phi(x) - \phi(x + \theta h) \), and \( K(x, h) \) and \( g(x, h) \) polynomials in \( h \) for fixed \( x \). The rest of the argument is the same as before.

The conclusion (20) can also be seen directly as follows: Differentiating (18) with respect to \( h \), we have

\[ \phi'(x + \theta h) \left( \theta + h \frac{\partial \theta}{\partial h} \right) = 0. \]

Making \( h \to 0 \) in this and noting that \( \theta(x, 0) = 1/(n + 1) \) in case (15b) we have \( \phi'(x) = 0 \), and hence \( \phi(x) = k \), a constant in \( (a, b) \).