PROOF OF A THEOREM OF SAKS AND SIERPINSKI

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The object of this note is to give a direct proof of the following theorem of Saks and Sierpinski:

If $f(x)$ is an arbitrary one-valued real function defined on the closed interval $I=[0, 1]$, there is a $\phi(x)$ of Baire class 2 at most such that for every $\epsilon>0$ the inequality $|f(x)-\phi(x)|<\epsilon$ holds on a set of exterior measure 1.

The proofs in the literature [1, 2][1] depend on the corresponding theorem for measurable functions; namely, if $f(x)$ is measurable, there is a $\phi(x)$ of Baire class 2 at most such that $f(x)=\phi(x)$ almost everywhere.

We first prove a lemma which seems to be new.

LEMMA. *If $f(x)$ is defined on the closed interval $I=[0, 1]$, $\epsilon>0$, $\phi(x)$ continuous, and $|f(x)-\phi(x)|<\epsilon$ on a set of exterior measure greater than $1-\epsilon$, then for every $\eta>0$ there is a continuous $\psi(x)$ such that $|\phi(x)-\psi(x)|<\epsilon$ on a set of measure greater than $1-\epsilon$ and $|f(x)-\psi(x)|$ $<\eta$ on a set of exterior measure greater than $1-\eta$.*

PROOF. $f(x)$ is exteriorly approximately continuous almost everywhere in the sense that for almost every $\xi\in I$ the set of points x for which $f(\xi)-k<f(x)<f(\xi)+k$ has exterior metric density 1 at ξ, for every $k>0$. There is a δ, with $0<\delta<\eta$, such that $|f(x)-\phi(x)|<\epsilon-\delta$ on a set E of exterior measure greater than $1-\epsilon$. Since $f(x)$ is exteriorly approximately continuous almost everywhere, every $\xi\in E-Z$, with Z of measure zero, is in a sequence, $\{I_{\xi n}\}$, $i=1, 2, \cdots$, of closed intervals, whose lengths converge to zero, such that the set of points x for which $|f(\xi)-f(x)<\delta$ has relative exterior measure exceeding $1-\delta/2$ in each $I_{\xi n}$. Moreover, since $\phi(x)$ is continuous, the $I_{\xi n}$ may be chosen so that the saltus of $\phi(x)$ in $I_{\xi n}$ is less than δ for every n. Consider the totality of intervals

$$I=[I_{\xi n}], \quad \xi\in E-Z, n=1, 2, \cdots .$$

By Vitali's covering theorem, since $m_e(E-Z)>1-\epsilon$, there is a finite number $I_{\xi_1 n_1}, I_{\xi_2 n_2}, \cdots, I_{\xi_k n_k}$ of disjoint intervals of I the sum, $\sum_{i=1}^{k} m(I_{\xi_i n_i})$, of whose lengths is $1-\alpha>1-\epsilon$. Let $G=I-\sum_{i=1}^{k} I_{\xi_i n_i}$ and let $G'\subset G$ be the points of G at which $f(x)$ is exteriorly approxi-

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[1] Numbers in brackets refer to the references cited at the end of the paper.

mately continuous. Every $\zeta \in G'$ is in a sequence $\{I_{\zeta m}\}$ of closed intervals contained in G, whose lengths converge to zero, such that for every m, the set of points x for which $|f(\zeta)-f(x)|<\delta$ has relative exterior measure in $I_{\zeta m}$ exceeding $1-\delta/2$. Consider the totality of intervals

$$J = [I_{\zeta m}], \qquad \zeta \in G', m = 1, 2, \cdots .$$

There is then a finite number of disjoint intervals $I_{\zeta_1 m_1}, I_{\zeta_2 m_2}, \cdots, I_{\zeta_K m_K}$ of J such that $\sum_{i=1}^{K} m(I_{\zeta_i m_i})$ exceeds $\alpha-\delta/2$. Now, let

$$\psi(x) = f(\xi_i) \qquad \text{for every } x \in I_{\xi_i n_i},\ i = 1, 2, \cdots, k,$$
$$\psi(x) = f(\zeta_i) \qquad \text{for every } x \in I_{\zeta_i m_i},\ i = 1, 2, \cdots, K,$$

and let $\psi(x)$ be defined elsewhere so as to be continuous. Suppose $x \in I_{\xi_i n_i}$, $1 \leqq i \leqq k$.

$$\begin{aligned} |\psi(x) - \phi(x)| &\leqq |\psi(x) - f(\xi_i)| + |f(\xi_i) - \phi(\xi_i)| \\ &\quad + |\phi(\xi_i) - \phi(x)| < 0 + (\epsilon - \delta) + \delta = \epsilon. \end{aligned}$$

But $\sum_{i=1}^{k} m(I_{\xi_i n_i}) > 1-\epsilon$. Hence $|\psi(x)-\phi(x)|<\epsilon$ on a set of measure greater than $1-\epsilon$. Moreover, since the relative exterior measure of the set of points for which $|\psi(x)-f(x)|<\delta<\eta$ exceeds $1-\delta/2$ in every $I_{\xi_i n_i}$, $i=1, 2, \cdots, k$, and in every $I_{\zeta_i m_i}$, $i=1, 2, \cdots, K$, and these intervals are disjoint, the exterior measure of this set exceeds

$$\begin{aligned} &\left(1 - \frac{\delta}{2}\right)\left[\sum_{i=1}^{k} m(I_{\xi_i n_i}) + \sum_{i=1}^{K} m(I_{\zeta_i m_i})\right] \\ &\qquad \geqq \left(1 - \frac{\delta}{2}\right)\left[(1 - \alpha) + \left(\alpha - \frac{\delta}{2}\right)\right] \\ &\qquad = \left(1 - \frac{\delta}{2}\right)^2 > 1 - \delta > 1 - \eta. \end{aligned}$$

This completes the proof.

We now prove the theorem of Saks and Sierpinski. Let $f(x)$ be an arbitrary function defined on $I=[0, 1]$. Let $\phi_0(x)\equiv 0$. By the lemma, there is a continuous $\phi_1(x)$ such that $|\phi_1(x)-f(x)|<1/2$ on a set of exterior measure greater than $1/2$. Having defined the continuous functions

$$\phi_1(x), \phi_2(x), \cdots, \phi_{n-1}(x)$$

with $|\phi_{n-1}(x)-f(x)|<1/2^{n-1}$ on a set of exterior measure greater than $1-1/2^{n-1}$ there is, by the lemma, a continuous $\phi_n(x)$ such that $|\phi_n(x)-\phi_{n-1}(x)|<1/2^{n-1}$ on a set of measure greater than $1-1/2^{n-1}$

and $|\phi_n(x)-f(x)|<1/2^n$ on a set of exterior measure greater than $1-1/2^n$. Let

$$\phi(x) = \limsup_{n\to\infty} \phi_n(x).$$

Then $\phi(x)=\lim_{n\to\infty}$ [l.u.b. $(\phi_n(x), \phi_{n+1}(x), \cdots)$], and, since l.u.b. $(\phi_n(x), \phi_{n+1}(x), \cdots)$ is lower semi-continuous, for every n, $\phi(x)$ is the limit of a nonincreasing sequence of lower semi-continuous functions and is, therefore, of Baire class 2 at most. Moreover, the sequence $\{\phi_n(x)\}$ itself converges almost everywhere to $\phi(x)$. Accordingly, for every $\epsilon>0$ and every $\eta>0$, there is an n, such that $|\phi_n(x)-\phi(x)|<\epsilon/2$ on a set of measure greater than $1-\eta/2$ and $|\phi_n(x)-f(x)|<\epsilon/2$ on a set of exterior measure greater than $1-\eta/2$. Hence, the set of points for which $|\phi(x)-f(x)|<\epsilon$ is of exterior measure greater than $1-\eta$ for every $\eta>0$ and is, therefore, of exterior measure one.

References

1. S. Saks and W. Sierpinski, *Sur une propriété générale de fonctions*, Fund. Math. vol. 11 (1928) pp. 105–112.

2. H. Blumberg, *The measurable boundaries of an arbitrary function*, Acta Math. vol. 65 (1935) p. 277.

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