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A ZERO-DIMENSIONAL TOPOLOGICAL GROUP WITH A ONE-DIMENSIONAL FACTOR GROUP

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As can be easily shown, if a locally compact topological group is zero-dimensional, all of its factor groups are zero-dimensional. In this note we give an example of a non locally compact zero-dimensional group with a factor group which is topologically isomorphic to the real numbers, hence one-dimensional.¹

1. **Preliminaries.** Let $\{\lambda\}$ be a set of indices of cardinality c , and for each λ , let R_λ be a topological isomorph of the additive group of rational numbers. We form the weak product R of the R_λ : an element r of R is a collection $r = \{r_\lambda\}$, $r_\lambda \in R_\lambda$, such that for only a finite number of the λ 's is $r_\lambda \neq 0_\lambda$. Under the definitions $r + r' = \{r_\lambda + r'_\lambda\}$, $0 = \{0_\lambda\}$, R forms a group.

Now for each $r \in R$, we define $\|r\| = \sum_\lambda |r_\lambda|$. Since all but a finite number of the $r_\lambda = 0_\lambda$, this sum exists. Clearly $\|r + r'\| \leq \|r\| + \|r'\|$, and $\|-r\| = \|r\|$, hence, as can be easily shown, $\|r\|$ defines a metric in R under the definition: the distance from r to r' is $\|r - r'\|$.

LEMMA 1. *Let $\{d_\lambda\}$ be a set of positive real numbers bounded away from zero, that is, there exists $d > 0$ such that $d_\lambda \geq d$ for all λ . Then*

$$U = \left\{ r \mid \sum_\lambda \left| \frac{r_\lambda}{d_\lambda} \right| < 1 \right\}$$

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¹ Cf. Bourbaki, *Topologie generale*, chap. III, p. 21, exercise 12, for an example of a totally disconnected group with a factor group topologically isomorphic to the reals. This example was pointed out to me by I. Kaplansky.

is an open set containing the origin, and

$$\bar{U} \subset \left\{ r \mid \sum_{\lambda} \left| \frac{r_{\lambda}}{d_{\lambda}} \right| \leq 1 \right\}.$$

PROOF. We need only prove that the real valued function $f(r) = \sum_{\lambda} |r_{\lambda}/d_{\lambda}|$ is continuous. Given $r = \{r_{\lambda}\}$ and $\epsilon > 0$, choose $\delta < d\epsilon$. Now, for any $r' = \{r'_{\lambda}\}$,

$$\begin{aligned} \left| \sum_{\lambda} \left| \frac{r'_{\lambda}}{d_{\lambda}} \right| - \sum_{\lambda} \left| \frac{r_{\lambda}}{d_{\lambda}} \right| \right| &= \left| \sum_{\lambda} \frac{|r'_{\lambda}| - |r_{\lambda}|}{d_{\lambda}} \right| \leq \sum_{\lambda} \left| \frac{|r'_{\lambda}| - |r_{\lambda}|}{d_{\lambda}} \right| \\ &\leq \frac{1}{d} \sum_{\lambda} ||r'_{\lambda}| - |r_{\lambda}|| \\ &\leq \frac{1}{d} \sum_{\lambda} |r'_{\lambda} - r_{\lambda}| \\ &= \frac{1}{d} \|r' - r\|. \end{aligned}$$

Thus, if $\|r' - r\| < \delta$, this last expression is less than ϵ , which proves the continuity and hence the lemma.

LEMMA 2. Let $\{\alpha_{\lambda}\}$ be a bounded set of positive irrational numbers linearly independent with respect to the rationals, that is:

$$(1) \quad a_1\alpha_{\lambda(1)} + \cdots + a_h\alpha_{\lambda(h)} = a \quad (a_1, \cdots, a_h, a \text{ rational})$$

implies

$$a_1 = \cdots = a_h = a = 0.$$

Then if, in Lemma 1, we take $d_{\lambda} = 1/\alpha_{\lambda}$, the resulting U has a vacuous boundary.

PROOF. From the second part of Lemma 1, we need only prove that there is no r such that $\sum_{\lambda} |r_{\lambda}/d_{\lambda}| = 1$. Assume there is. Then, since replacing an r_{λ} by its negative does not change absolute values, we can assume all the r_{λ} are non-negative. Then we have $\sum_{\lambda} r_{\lambda}/d_{\lambda} = 1$. But each $d_{\lambda} = 1/\alpha_{\lambda}$, hence $\sum_{\lambda} r_{\lambda}\alpha_{\lambda} = 1$, which contradicts the hypothesis on the α_{λ} 's.

Using Lemma 2, we now define a special sequence of neighborhoods $\{U_n\}$ ($n = 0, 1, \cdots$) which form a basis around the origin. We first take the set of real numbers $1/2 \leq c < 1$ and set them in one-one correspondence with the λ 's: $\{c_{\lambda}\}$. (Our purpose in bringing these in will

become clear in §2.) We then choose a set of irrational numbers $\{\alpha_\lambda\}$ with the property (1) (the existence of such a set follows from the existence of a Hamel basis for the reals), and such that for each λ , $c_\lambda < \alpha_\lambda < 1$. This last can always be accomplished by multiplying α_λ by a suitable rational. We then have

$$(2) \quad 1/2 \leq c_\lambda < \alpha_\lambda < 1 \quad \text{for all } \lambda.$$

Now for each n , we define U_n by taking

$$(3) \quad d_\lambda^{(n)} = \frac{1}{2^n} \cdot \frac{1}{\alpha_\lambda},$$

and letting

$$(4) \quad U_n = \left\{ r \mid \sum_\lambda \left| \frac{r_\lambda}{d_\lambda^{(n)}} \right| < 1 \right\}.$$

Since, for $r \in U_n$,

$$\|r\| = \sum_\lambda |r_\lambda| = \sum_\lambda d_\lambda^{(n)} \left| \frac{r_\lambda}{d_\lambda^{(n)}} \right| \leq \frac{2}{2^n} \sum_\lambda \left| \frac{r_\lambda}{d_\lambda^{(n)}} \right| < \frac{1}{2^{n-1}},$$

the diameter of U_n is less than $1/2^{n-2}$, hence approaches zero as n goes to infinity. Thus $\{U_n\}$ constitutes a basis around the origin. Since, from Lemma 2, the boundary of each U_n is vacuous, it follows that R is zero-dimensional.

2. The example. Let R_* be the additive group of real numbers $\{r_*\}$ with distance defined by $\|r_*\| = 1$ for all r_* different from zero. This makes it discrete. Let $G = R_* \times R$ with distance defined as follows: If $g = (r_*, r)$ then $\|g\| = \|r_*\| + \|r\|$. Since R_* is discrete, the U_n 's, now considered as subsets of G , form a basis around the origin of G , hence G is zero-dimensional.

We define the subgroup H of G as the set of all $g = (r_*, r)$ such that $r_* + \sum_\lambda c_\lambda r_\lambda = 0$ (cf. (2)).

LEMMA 3. *H is a closed subgroup of G.*

PROOF. H is a subgroup, for if $g, g' \in H$ then $r_* + \sum_\lambda c_\lambda r_\lambda = 0$ and $r'_* + \sum_\lambda c_\lambda r'_\lambda = 0$, hence $(r_* - r'_*) + \sum_\lambda c_\lambda (r_\lambda - r'_\lambda) = 0$. To prove H is closed, it is sufficient to show that the real-valued function $f(g) = r_* + \sum_\lambda c_\lambda r_\lambda$ is continuous. Given $g = (r_*, r)$ and $\epsilon > 0$, choose $\delta < \min(\epsilon, 1)$. Consider any $g' = (r'_*, r')$ such that $\|g' - g\| < \delta$. We note first that $r'_* = r_*$, for otherwise $\|r'_* - r_*\| = 1$ and hence $\|g' - g\| \geq 1 \geq \delta$. Then

$$\begin{aligned}
\left| r'_* + \sum_{\lambda} c_{\lambda} r'_{\lambda} - r_* - \sum_{\lambda} c_{\lambda} r_{\lambda} \right| &= \left| \sum_{\lambda} c_{\lambda} r'_{\lambda} - \sum_{\lambda} c_{\lambda} r_{\lambda} \right| \\
&\leq \sum_{\lambda} c_{\lambda} |r'_{\lambda} - r_{\lambda}| \\
&\leq \sum_{\lambda} |r'_{\lambda} - r_{\lambda}| \quad (\text{from (2)}) \\
&= \|g' - g\| < \delta < \epsilon.
\end{aligned}$$

This proves the lemma.

LEMMA 4. G/H is algebraically isomorphic to the real numbers.

PROOF. Since $R_* \times 0 \subset G$ is algebraically isomorphic to the real numbers, it is sufficient to prove that G/H is algebraically isomorphic to this subgroup.

(i) Every coset of H contains an element of $R_* \times 0$.

For, let $g = (r_*, r)$ be any element of G , where $r = \{r_{\lambda}\}$. Then if $g' = (r_* + \sum_{\lambda} c_{\lambda} r_{\lambda}, 0)$, $g' - g = (\sum_{\lambda} c_{\lambda} r_{\lambda}, -r) \in H$. Since $g' \in R_* \times 0$, this proves (i).

(ii) Different elements of $R_* \times 0$ lie in different cosets.

For, let $g = (r_*, 0)$, $g' = (r'_*, 0)$, with $r_* \neq r'_*$. Then $g' - g = (r'_* - r_*, 0)$, and since $(r'_* - r_*) + 0 \neq 0$, $g' - g \notin H$. This proves (ii), and with it, Lemma 4.

We can thus denote each element of G/H by a unique real number. From the proof of (i) above, we see that the real number is given by the mapping

$$(5) \quad \pi(r_*, r) = r_* + \sum_{\lambda} c_{\lambda} r_{\lambda}.$$

LEMMA 5. $\pi(U_n) = [-1/2^n < x < 1/2^n]$ for all $n = 0, 1, \dots$.

PROOF. Since the argument is the same for all n , it is sufficient to prove this for U_0 . From (4) and (3),

$$U_0 = \left\{ g = (0, r) \mid \sum_{\lambda} |c_{\lambda} r_{\lambda}| < 1 \right\}.$$

(i) $\pi(U_0) \subset [-1 < x < 1]$.

For, if $g = (0, r) \in U_0$, then from (5),

$$\begin{aligned}
|\pi(g)| &= \left| \sum_{\lambda} c_{\lambda} r_{\lambda} \right| \leq \sum_{\lambda} |c_{\lambda} r_{\lambda}| \\
&< \sum_{\lambda} |c_{\lambda} r_{\lambda}| \quad (\text{from (2)}) \\
&< 1.
\end{aligned}$$

(ii) $[-1 < x < 1] \subset \pi(U_0)$.

For consider any real number x such that $-1 < x < 1$. Since the c_λ 's run through all the real numbers from $1/2$ to 1 , there is a c_λ such that

$$x = \epsilon c_\lambda,$$

where ϵ is one of the values $\pm 1, \pm 1/2$. Hence, if g is the element of G whose λ '-coordinate is ϵ and whose remaining coordinates are 0 , we have from (5) that

$$\pi(g) = c_\lambda \epsilon = x.$$

Thus x has an inverse in U_0 under π . Since x was any element of $[-1 < x < 1]$, (ii) is proved. This establishes the lemma.

Since the set $\{U_n\}$ is a basis around the origin of G , the set $\{\pi U_n\}$ is by definition a basis around zero in G/H . Hence, from Lemma 5, G/H has the topology of the real numbers.

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