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A CONJECTURE OF KRISHNASWAMI

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Let $T(N)$ denote the number of right triangles whose perimeters do not exceed $2N$, and whose sides are relatively prime integers. A list of all such triangles whose perimeters do not exceed 10000 has been given by A. A. Krishnaswami.¹ On the basis of this table he conjectured that

$$(1) \quad T(N) \sim N/7.$$

The asymptotic formula

$$(2) \quad T(N) \sim \pi^{-2}N \log 4$$

follows from the general theory of “totient points,” as developed by D. N. Lehmer in 1900. A statement equivalent to (2) will be found in his paper² (p. 328).

The conjecture (1) is not far wrong since

$$\pi^2/\log 4 = 7.11941466.$$

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¹ A. A. Krishnaswami, *On isoperimetrical Pythagorean triangles*, Tôhoku Math. J. vol. 27 (1926) pp. 332–348. Two omissions in Table I may be noted: For $s=3450$, $a=50$, $b=19$; for $s=3465$, $a=55$, $b=8$. This table is the basis for the one at the end of the present paper.

² D. N. Lehmer, *Asymptotic evaluation of certain totient sums*, Amer. J. Math. vol. 22 (1900) pp. 293–335.

In this paper we give a short proof of the fact that

$$(3) \quad T(N) = \pi^{-2}N \log 4 + O(N^{1/2} \log N).$$

The actual values of the error term for $N = 500(500)5000$ are given in a short table at the end of this paper. The proof of (3) is based on the following lemmas.

LEMMA 1. Let $\phi(m)$ denote the number of positive integers not exceeding m and prime to m . Then

$$(4) \quad \Phi(x) \equiv \sum_{m \leq x} \phi(m) = 3\pi^{-2}x^2 + O(x \log x).$$

A proof of this well known result will be found for example in Hardy and Wright³ (p. 266).

LEMMA 2. Let $\Phi_e(x)$ and $\Phi_o(x)$ be defined by

$$\Phi_e(x) = \sum_{m \leq x, m \text{ even}} \phi(m), \quad \Phi_o(x) = \sum_{m \leq x, m \text{ odd}} \phi(m).$$

Then

$$(5) \quad \Phi_e(x) = \pi^{-2}x^2 + O(x \log x),$$

$$(6) \quad \Phi_o(x) = 2\pi^{-2}x^2 + O(x \log x).$$

PROOF. Since (6) follows from (4) and (5) it suffices to prove (5). To this effect we note that if m is even

$$(7) \quad \phi(m) = \begin{cases} \phi(m/2), & m \equiv 2 \pmod{4}, \\ 2\phi(m/2), & m \equiv 0 \pmod{4}. \end{cases}$$

Hence

$$\Phi_e(x) = \Phi_o(x/2) + 2\Phi_e(x/2) = \Phi(x/2) + \Phi_o(x/2).$$

Therefore

$$\Phi_e(x) = \sum_{\lambda=1}^p \Phi(2^{-\lambda}x) \quad (p = [\log x / \log 2]).$$

Applying Lemma 1 we have

$$\Phi_e(x) = 3\pi^{-2}x^2 \sum_{\lambda=1}^p 4^{-\lambda} + O(x \log x)$$

³ G. H. Hardy and E. M. Wright, *Introduction to the theory of numbers*, Oxford, 1938. Lemma 1 appears to be due to Mertens, *Journal für Mathematik* vol. 77 (1871) pp. 289–291.

$$= \pi^{-2}x^2 + O\left(x^2 \int_p^\infty 4^{-t} dt\right) + O(x \log x).$$

Since $p > \log x / \log 4$, the integral is $O(1/x)$. Hence (5) follows.

LEMMA 3. Let $0 < \theta < 1$, and define $F(\theta, x)$, $F_e(\theta, x)$ and $F_o(\theta, x)$ by

$$F(\theta, x) = \sum_{\theta x < m \leq x} m^{-2} \phi(m), \quad F_e(\theta, x) = \sum_{\theta x < m \leq x, m \text{ even}} m^{-2} \phi(m),$$

$$F_o(\theta, x) = \sum_{\theta x < m \leq x, m \text{ odd}} m^{-2} \phi(m).$$

Then as $x \rightarrow \infty$, with θ fixed,

(8) $F(\theta, x) = -6\pi^{-2} \log \theta + O(x^{-1} \log x),$

(9) $F_e(\theta, x) = -2\pi^{-2} \log \theta + O(x^{-1} \log x),$

(10) $F_o(\theta, x) = -4\pi^{-2} \log \theta + O(x^{-1} \log x).$

PROOF. Since (10) follows from (8) and (9) it suffices to prove (8) and (9). Now

$$F(\theta, x) = \sum_{\theta x < m \leq x} m^{-2} \phi(m) = \sum_{\theta x < m \leq x} \{ \Phi(m) - \Phi(m-1) \} m^{-2}$$

$$= \sum_{\theta x < m \leq x} \Phi(m) \{ m^{-2} - (m+1)^{-2} \}$$

$$- \Phi(\theta x) [\theta x + 1]^{-2} + \Phi(x) [x + 1]^{-2}.$$

By Lemma 1 these last two terms cancel to some extent and together contribute only $O(x^{-1} \log x)$. As for the rest

$$\sum_{\theta x < m \leq x} \Phi(m) \{ m^{-2} - (m+1)^{-2} \}$$

$$= 3\pi^{-2} \sum (1 - (1 + m^{-1})^{-2}) + O(\sum m^{-1} (1 - (1 + m^{-1})^{-2}) \log m)$$

$$= 3\pi^{-2} \sum 2m^{-1} (1 + O(m^{-1})) + O(\sum m^{-2} \log m)$$

$$= 6\pi^{-2} \int_{\theta x}^x t^{-1} dt + O(x^{-1}) + O\left(\int_{\theta x}^x t^{-2} \log t dt\right)$$

$$= -6\pi^{-2} \log \theta + O(x^{-1} \log x),$$

which gives (8). To prove (9) we note from (7) that

$$F_e(\theta, x) = F_o(\theta, x/2)/4 + F_e(\theta, x/2)/2 = F(\theta, x/2)/4 + F_e(\theta, x/2)/4.$$

Hence

$$F_e(\theta, x) = \sum_{\lambda=1}^p F(\theta, x/2^\lambda) 4^{-\lambda} \quad (p = [\log x / \log 2]).$$

Using (8) we find

$$F_s(\theta, x) = -6\pi^{-2} \log \theta \sum_{\lambda=1}^{\infty} 4^{-\lambda} + O\left(\int_p^{\infty} 4^{-t} dt\right) + O(x^{-1} \log x).$$

Since the integral is $O(x^{-1})$, (9) follows at once. This completes the proof of Lemma 3.

LEMMA 4. *Let $\phi(x, m)$ denote the number of integers $\leq x$ and prime to m . Then*

$$|\phi(x, m) - xm^{-1}\phi(m)| < d(m)$$

where $d(m)$ is the number of divisors of m .

This follows easily from a familiar theorem of Legendre to the effect that

$$(11) \quad \phi(x, m) = \sum_{\delta|m} [x\delta^{-1}]\mu(\delta)$$

where μ is the Möbius function and the sum extends over all the divisors of m . In fact if we write

$$[x\delta^{-1}] = x\delta^{-1} - \epsilon(x, \delta)$$

so that

$$0 \leq \epsilon(x, \delta) < 1,$$

then (11) becomes

$$\phi(x, m) = x \sum \delta^{-1}\mu(\delta) - \sum \epsilon(x, \delta)\mu(\delta).$$

The first sum is $m^{-1}\phi(m)$ and the second is less than

$$\sum_{\delta|m} 1 = d(m)$$

in absolute value. This proves the lemma.

Finally we need one more lemma.

LEMMA 5.

$$\sum_{m \leq x} d(m) = O(x \log x).$$

This is a very weak corollary of a famous result of Dirichlet (see Hardy and Wright,³ p. 262–263).

We are now in a position to prove the following theorem.

THEOREM. *Let $T(N)$ denote the number of integral right triangles whose perimeters do not exceed $2N$ and whose sides are relatively prime, then*

$$T(N) = \pi^{-2}N \log 4 + O(N^{1/2} \log N).$$

PROOF. It is well known that all integral right triangles (a, b, c) are given by the parametric equations

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2$$

where m, n are integers with

$$(12) \quad n \leq m.$$

Since the perimeter is supposed not to exceed $2N$ we have

$$(13) \quad mn + m^2 \leq N.$$

In order to avoid the cases in which a, b, c have a common factor it is necessary to suppose that we choose m, n so that

$$(14) \quad m, n \text{ are coprime and not both odd.}$$

$T(N)$ is then merely the number of pairs of positive integers (m, n) such that (12), (13) and (14) hold. In case $m \leq (N/2)^{1/2}$, (13) is a consequence of (12). In case $(N/2)^{1/2} < m \leq N^{1/2}$, (12) is a consequence of (13). Hence if we define

$$(15) \quad \psi(m) = \begin{cases} 1 & \text{if } m \leq (N/2)^{1/2}, \\ m^{-2}N - 1 & \text{if } (N/2)^{1/2} < m \leq N^{1/2} \end{cases}$$

then the number of integers n that go with a given m is the number of integers prime to m not exceeding $m\psi(m)$ or $m\psi(m)/2$ according as m is even or odd. Hence if we set $x = N^{1/2}$

$$T(N) = \sum_{m \leq x, m \text{ even}} \phi(m\psi(m), m) + \sum_{m \leq x, m \text{ odd}} \phi\left(\frac{m\psi(m)}{2}, m\right).$$

By Lemma 4,

$$(16) \quad \begin{aligned} T(N) &= \sum_{m \leq x, m \text{ even}} \psi(m)\phi(m) + 2^{-1} \sum_{m \leq x, m \text{ odd}} \psi(m)\phi(m) + R(N) \\ &= \sum_1 + 2^{-1} \sum_2 + R(N) \end{aligned}$$

where

$$|R(N)| \leq \sum_{m \leq x} d(m) = O(x \log x) = O(N^{1/2} \log N).$$

By (15) with $\theta = 2^{-1/2}$ we can write

$$\sum_1 = \Phi_e(\theta x) + NF_e(\theta, x) - \Phi_e(x) + \Phi_e(\theta x),$$

$$\sum_2 = \Phi_0(\theta x) + NF_0(\theta, x) - \Phi_0(x) + \Phi_0(\theta x),$$

so that

$$\begin{aligned} \sum_1 + 2^{-1} \sum_2 &= \Phi(\theta x) + \Phi_e(\theta x) - 2^{-1} \{ \Phi(x) + \Phi_e(x) \} \\ &\quad + 2^{-1} N \{ F(\theta, x) + F_e(\theta, x) \}. \end{aligned}$$

By Lemmas 1, 2, and 3 therefore we obtain after simplification

$$T(N) = \pi^{-2}(2 \log 2)x^2 + O(x \log x) = \pi^{-2}(\log 4)N + O(N^{1/2} \log N).$$

The following small table illustrates the error in (3):

$$E(N) = T(N) - \pi^{-2}N \log 4.$$

The function $C(N)$ is defined by

$$C(N)N^{1/2} \log N = 10^3 E(N)$$

and gives some idea of the possible constant implied by the O term of (3).

N	$T(N)$	ΔT	$\pi^{-2} N \log 4$	$E(N)$	$C(N)$
500	70	70	70.23049	-0.23049	-1.6596
1000	140	71	140.46099	-0.46099	-2.1103
1500	211	69	210.69148	+0.30852	+1.0893
2000	280	69	280.92197	-0.92197	-2.7123
2500	349	73	351.15246	-2.15246	-5.5022
3000	422	70	421.38296	-0.61704	-1.4071
3500	492	68	491.61345	+0.38655	+0.8007
4000	560	71	561.84394	-1.84394	-3.5152
4500	631	72	632.07444	-1.07444	-1.9041
5000	703		702.30493	+0.69507	+1.1541

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