ON THE CHARACTERISTIC EQUATIONS OF CERTAIN MATRICES

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In a recent paper Brauer[1] proved the following theorem credited to R. v. Mises.

THEOREM. *Let* $A=(a_{ij})$, $B=(b_{ij})$, *and* $C=(c_{ij})$ *be square matrices of order* n. *If the elements of* A *and* C *satisfy the conditions*

$$r_i = \sum_{j=1}^{n} a_{ij} = 0 \qquad (i = 1, 2, \cdots, n), \tag{1}$$

$$s_j = \sum_{i=1}^{n} a_{ij} = 0 \qquad (j = 1, 2, \cdots, n), \tag{2}$$

$$c_{ij} = c_i + c_j \qquad (i, j = 1, 2, \cdots, n), \tag{3}$$

where $c_1, c_2, \cdots, c_n$ *are arbitrary numbers, then the matrices* AB *and* $A(B+C)$ *have the same characteristic equation.*

Write $C_1=c'e$ where $c=(c_1, c_2, \cdots, c_n)$ and $e=(1, 1, \cdots, 1)$ then conditions (1), (2), and (3) are $AC_1'=0$, $C_1A=0$ and $C=C_1+C_1'$. This is a special case of the following theorem.

THEOREM. *Let* A, C_1, *and* C_2 *be* n*-rowed square matrices such that* $C_1A=AC_2=0$. *If* $C=C_1+C_2$ *and* B *is an arbitrary* n*-rowed square matrix, then* AB *and* $A(B+C)$ *have the same characteristic equation.*

The theorem is trivial if A is nonsingular, for then $C=0$. The proof will be based on the well known lemma:

LEMMA. *If* A *and* B *are square matrices,* AB *and* BA *have the same characteristic equation.*

Since $AC_2=0$, $A(B+C)=A(B+C_1)$ and from the lemma it follows that $A(B+C_1)$ has the same characteristic equation as $(B+C_1)A=BA$, and BA has the same characteristic equation as AB.

It may be readily shown that if A and C are matrices (not necessarily square) such that $ACA=0$, then $C=C_1+C_2$ where $AC_2=C_1A=0$. Also if A is an $m\times n$ matrix and B and C are $n\times m$ matrices and $ACA=0$, there exists a nonsingular matrix P, such that

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[1] Alfred Brauer, *On the characteristic equations of certain matrices*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 605–607.

$$PABP^{-1} = \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad PACP^{-1} = \begin{pmatrix} 0 & C_2 \\ 0 & 0 \end{pmatrix}$$

and hence in this case AB and $A(B+C)$ have the same characteristic equation.

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