ON THE EXTENSION OF A TRANSFORMATION

EARL J. MICKLE

0. Introduction. In a problem on surface area the writer and Helsel\(^1\) were confronted with the following question. Can a Lipschitzian transformation from a set in a Euclidean three-space into a Euclidean three-space be extended to a Lipschitzian transformation defined on the whole space? The affirmative answer to this question has been given by Kirszbraun.\(^2\) In fact, Kirszbraun shows this result for any Euclidean spaces (see also Valentine).\(^3\) In studying these papers the writer noted that a more general extension problem could be formulated and a different method of proof to the problem could be obtained. To formulate the more general problem we first give some definitions.

Let \(M\) be a metric space, the distance between two points \(p_1, p_2 \in M\) being denoted by \(d(p_1, p_2)\). Let \(\mathcal{P}(M)\) be the class of real-valued continuous functions \(g(t), 0 \leq t < \infty\), which satisfy the conditions: (a) \(g(0) = 0\), (b) \(g(t) > 0\) for \(t > 0\), (c) for any finite number of points \(p_0, p_1, \ldots, p_m\) in \(M\) the real quadratic form \(\sum_{i,j=1}^{m} [g(p_i p_j)^2 + g(p_i p_j)^2 - g(p_i p_j)^2] \xi_i \xi_j\) is positive. Let \(g(t) \in \mathcal{P}(M)\). A transformation \(p^* = \phi(p)\) from a set \(E\) in \(M\) into a metric space \(M^*\) will be said to satisfy the condition \(C(g)\) on \(E\) if, for every pair of points \(p, p^* \in E\), \(d(p^*, p^*) \leq g(d(p, p^*))\), where \(p^* = \phi(p)\), \(i = 1, 2\). We shall say that \(\phi(p)\) can be extended to a set \(E'\), \(E \subset E' \subset M\), preserving the condition \(C(g)\) if there exists a transformation \(p^* = \Phi(p)\) from \(E'\) into \(M^*\) which satisfies the condition \(C(g)\) on \(E'\) and is equal to \(\phi(p)\) on \(E\).

In this paper we prove the following result. Let \(M\) be a separable metric space and let \(g(t) \in \mathcal{P}(M)\). Then any transformation from a set \(E\) in \(M\) into a Euclidean space which satisfies the condition \(C(g)\) on \(E\) can be extended to \(M\) preserving the condition \(C(g)\).

We give two examples to illustrate this result. We shall use the vector notation \(x\) to represent a point in a Euclidean \(n\)-space \(E_n\), and we shall denote by \(|x_1 - x_2|\) the distance between two points \(x_1, x_2\). Let \(x_0, x_1, \ldots, x_m\) be \(m + 1\) points in \(E_n\) and let \(\xi_1, \ldots, \xi_m\) be \(m\) real numbers. From the relation \((x_i - x_j)^2 = (x_0 - x_i)^2 + (x_0 - x_j)^2 - 2(x_0 - x_i)(x_0 - x_j)\), the square of the vector \(x = \sum \xi_i (x_0 - x_i) + \cdots\)

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$+L\xi_m(x_0-x_m)$, $L>0$, is given by

$$x^2 = L^2 \sum_{i,j=1}^{m} (x_0 - x_i)(x_0 - x_j)\xi_i\xi_j$$

$$= \frac{L^2}{2} \sum_{i,j=1}^{m} \left[ |x_0 - x_i|^2 + |x_0 - x_j|^2 - |x_i - x_j|^2 \right] \xi_i \xi_j \geq 0.$$

Thus the function $g(t) = Lt$, $L>0$, is in $P(E_n)$ and the results of Kirszbraun follow as a special case of the results of this paper. Schoenberg\(^\text{4}\) has shown that the function $g(t) = L^\alpha$, $L>0$, $0<\alpha \leq 1$, is in $P(E_n)$. Thus a transformation from a set in a Euclidean space into a Euclidean space which satisfies a Lipschitz-Hölder condition ($L>0$, $0<\alpha \leq 1$) can be extended to the whole space preserving this same Lipschitz-Hölder condition.

1. Preliminary remarks. In this section we give some well known concepts and lemmas for a Euclidean $n$-space $E_n$. A set is called convex if the line segment joining any two points of the set is in the set. If $E$ is a closed convex set and $x \in E$, then there is a unique point $x^* \in E$ which is closest to $x$. (Since $E$ is closed there is one such point. If there were two, the midpoint of the line segment joining them would be in $E$ and closer to $x$ than either of them.) For a finite set of points $x_1, \ldots, x_m$, we denote by $V(x_1, \ldots, x_m)$ the smallest convex set containing them. $V(x_1, \ldots, x_m)$ is a closed set consisting of those points given by the relation $x = c_1x_1 + \cdots + c_mx_m$, where the $c_i$'s are non-negative and $c_1 + \cdots + c_m = 1$.

**Lemma 1.1.** Let $E$ be a closed convex set, $x_0$ a point not in $E$, $x_0^*$ the unique point of $E$ closest to $x_0$, and $y = tx_0 + (1-t)x_0^*$, $0 \leq t < 1$. Then $|y-x| < |x_0-x|$ for every point $x \in E$.

**Proof.** Since $|y-x| \leq t|x_0-x| + (1-t)|x_0^*-x|$, $t \neq 1$, it is sufficient to prove that $|x_0^*-x| < |x_0-x|$ for all $x \in E$. Assume there is a point $x_1 \in E$ for which $|x_0^*-x_1| \geq |x_0-x_1|$. Then the numbers $a = |x_0-x_1|$, $b = |x_0-x_1|$, $c = |x_0^*-x_1|$ satisfy the inequalities $a < b \leq c$ and the number $t^* = (a^2+c^2-b^2)/2c^2$ satisfies the inequalities $0 < t^* < 1/2$. Thus the point $x = t^*x_1 + (1-t^*)x_0^*$ is in $E$ and $|x_0-x|^2 = |x_0-x_0^*|^2 + t^*2c^2 < a^2$, contradicting the assumption that $x_0^*$ is the point of $E$ closest to $x_0$.

**Lemma 1.2.** Let $\Sigma$ be a set of closed spheres in $E_m$ such that there is no point in common to all of them. Then there is a finite set of spheres in $\Sigma$

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which have no point in common.\footnote{We use the term closed sphere to mean the set of points \( x \) which satisfy the inequality \( |x - x_0| \leq r \) for fixed \( x_0 \) and \( r > 0 \).}

**Proof.** Let \( S_0 \) be one of the spheres in \( \Sigma \). Then the sum of the complements of the remaining spheres cover \( S_0 \). Since this is an open covering of \( S_0 \), there is a finite number of spheres \( S_1, \ldots, S_m \) in \( \Sigma \) the sum of whose complements covers \( S_0 \). Hence, \( S_0, S_1, \ldots, S_m \) have no point in common.

2. **Lemmas.** For a given integer \( m \), let \( a_1, \ldots, a_m \) be a given set of \( m \) positive numbers and let \( x_1, \ldots, x_m \) be a given set of (not necessarily distinct) \( m \) points in \( E_n \). For each point \( x \) let

\[
f(x) = \max \left( \frac{|x - x_i|}{a_i} \right), \quad i = 1, \ldots, m.
\]

\( f(x) \) is obviously a continuous function of \( x \) and there exists a point \( x_0 \) for which \( f(x_0) = \min f(x) \). We have the following result concerning the location of a point \( x_0 \) at which \( f(x) \) assumes a minimum value.

**Lemma 2.1.** Let \( x_0 \) be a point such that \( f(x_0) = \min f(x) \) and let \( x_{m_1}, \ldots, x_{m_k} \) be all the points in the set of points \( x_1, \ldots, x_m \) for which the equality

\[
f(x_0) = \left| x_0 - x_i \right| / a_i
\]

holds. Then \( x_0 \in V(x_{m_1}, \ldots, x_{m_k}) \).

**Proof.** Assume \( x_0 \in V(x_{m_1}, \ldots, x_{m_k}) \). Let \( x_0^* \) be the unique point of \( V(x_{m_1}, \ldots, x_{m_k}) \) closest to \( x_0 \). For each integer \( j \) let \( y_j = t_j x_0 + (1 - t_j) x_0^* \), \( 0 \leq t_j < 1 \), \( t_j \to 1 \), for \( j \to \infty \). By Lemma 1.1, \( |y_j - x| < |x_0 - x| \) for every point \( x \in V(x_{m_1}, \ldots, x_{m_k}) \). Thus, since \( f(x_0) \leq f(y_j), f(y_j) = |y_j - x_i| / a_i, 1 \leq i \leq m \), for some \( x_i \in V(x_{m_1}, \ldots, x_{m_k}) \). There are an infinite number of the \( x_i \)'s corresponding to the points \( y_j \) which are the same and we can assume without loss of generality that all of them are the same point \( x_{m_{k+1}}, 1 \leq m_{k+1} \leq m \). Since \( f(y_j) \to f(x_0) \) for \( j \to \infty \), \( f(x_0) = \lim \frac{|y_j - x_{m_{k+1}}|}{a_{m_{k+1}}} = \frac{|x_0 - x_{m_{k+1}}|}{a_{m_{k+1}}} \), \( x_{m_{k+1}} \in V(x_{m_1}, \ldots, x_{m_k}) \). Thus (2.2) holds for \( i = m_{k+1} \). This contradicts the fact that (2.2) holds only for the points \( x_{m_1}, \ldots, x_{m_k} \). Hence \( x_0 \in V(x_{m_1}, \ldots, x_{m_k}) \).

We now prove the fundamental lemma of the paper.

**Lemma 2.2.** Let \( M \) be a metric space, let \( g(t) \in \mathcal{P}(M) \) and let \( p_1, \ldots, p_m \) be \( m \) distinct points and \( x_1, \ldots, x_m \) be \( m \) points in \( M \) and \( E_n \) respectively for which the inequalities \( |x_i - x_j| \leq g(p_i, p_j), i, j = 1, \ldots, m \), hold. Then, for any point \( p_0 \in M \), there exists a point
for which the inequalities \( |x_0 - x_i| \leq g(p_0p_i), \quad i = 1, \ldots, m, \) hold.

**Proof.** If \( p_0 = p_i, \quad 1 \leq i \leq m, \) let \( x_0 = x_i. \) Assume \( p_0 \neq p_1, \ldots, p_m. \) Set \( a_i = g(p_0p_i), \quad i = 1, \ldots, m. \) Since \( g(t) \in \mathcal{P}(M), \) each \( a_i > 0. \) If \( x_0 \) is a point for which \( f(x_0) = \min f(x) \) (see (2.1)), we assert that \( \lambda = f(x_0) \leq 1. \) That is to say, \( |x_0 - x_i| \leq g(p_0p_i), \quad i = 1, \ldots, m. \) If \( \lambda = 0, \) then \( \lambda \leq 1. \) Assume \( \lambda > 0. \) Set \( a_{ij} = g(p_ip_j), \quad b_{ij} = |x_i - x_j|, \) and \( b_i = |x_0 - x_i|, \) \( i, j = 1, \ldots, m. \) By renumbering if necessary, let \( x_1, \ldots, x_k \) be the points for which the equality \( f(x_0) = b_i/a_i = \lambda \) holds. By Lemma 2.1, \( x_0 \in V(x_1, \ldots, x_k). \) Thus we have non-negative numbers \( c_1, \ldots, c_k, \)

\[ c_1 + \cdots + c_k = 1 \] such that \( x_0 = c_1x_1 + \cdots + c_kx_k \) or \( c_1(x_0 - x_1) + \cdots + c_k(x_0 - x_k) = 0. \) By squaring this expression and using the relation \( (x_i - x_j)^2 = (x_0 - x_i)^2 + (x_0 - x_j)^2 - 2(x_0 - x_i)(x_0 - x_j) \) we obtain

\[
(2.3) \quad \sum_{i, j = 1}^k (x_0 - x_i)(x_0 - x_j)c_ic_j = \frac{1}{2} \sum_{i, j = 1}^k (b_i^2 + b_j^2 - b_{ij}^2)c_ic_j = 0.
\]

Since \( \lambda > 0, \) \( x_0 \neq x_1, \ldots, x_k, \) and hence at least two of the \( c_i \)’s are different from zero. Since \( g(t) \in \mathcal{P}(M), \) the quadratic form \( \sum_{i,j=1}^k (a_i^2 + a_j^2 - a_{ij}^2)c_ic_j \) is positive. Setting \( \xi_i = \lambda c_i, \quad i = 1, \ldots, k, \) and using the fact that \( \lambda a_i = b_i, \quad i = 1, \ldots, k, \) we obtain

\[
(2.4) \quad \frac{1}{2} \sum_{i, j = 1}^k (a_i^2 + a_j^2 - a_{ij}^2)\lambda^2 c_ic_j = \frac{1}{2} \sum_{i, j = 1}^k (b_i^2 + b_j^2 - \lambda^2 a_{ij}^2)c_ic_j \geq 0.
\]

Subtracting (2.3) from (2.4) gives

\[
(2.5) \quad \frac{1}{2} \sum_{i, j = 1}^k (b_{ij}^2 - \lambda^2 a_{ij}^2)c_ic_j \geq 0.
\]

Since \( a_{ij} = b_{ij} = 0 \) for \( i = j, \) the \( c_i \)’s are non-negative and at least two of the \( c_i \)’s are different from zero, it follows from (2.5) that \( b_{ij}^2 - \lambda^2 a_{ij}^2 \geq 0 \) for some pair of integers \( i, j \) with \( i \neq j. \) For this pair of integers \( 1 \leq i, j \leq k, \) \( i \neq j, \) we have \( g(p_ip_j)^2 \geq |x_i - x_j|^2 = b_{ij}^2 \geq \lambda^2 a_{ij}^2 = \lambda^2 g(p_ip_j)^2. \) Hence \( \lambda \leq 1. \) Thus a point \( x_0 \) at which \( f(x) \) assumes a minimum satisfies the conditions of the lemma.

**Lemma 2.3.** Let \( M \) be a metric space, let \( g(t) \in \mathcal{P}(M), \) let \( x = \phi(p) \) be a transformation from a set \( E \) in \( M \) into \( E, \) and let \( p_0 \) be any point in \( M. \) Then, if \( \phi(p) \) satisfies the condition \( C(g) \) on \( E, \phi(p) \) can be extended to \( E + p_0 \) preserving the condition \( C(g). \)

**Proof.** If \( p_0 \in E \) the extension is immediate. Assume \( p_0 \notin E. \) For each \( \bar{p} \in E, \) let \( S_\bar{p} \) be the set of points \( x \in E, \) which satisfy the in-
equality \( |x - \phi(p)| \leq g(p_0p) \). Since \( g(t) \in \mathcal{P}(M) \), each \( S_p \) is a closed sphere in \( E_n \). Assume that there is no point in common to all the spheres. By Lemma 1.2 there is a finite number of these spheres which have no point in common. This contradicts Lemma 2.2. Hence, there is at least one point \( x_0 \) in all the spheres \( S_p, \ p \in E \). Then \( \Phi(p_0) = x_0, \ \Phi(p) = \phi(p), \ p \in E \), is an extension of \( \phi(p) \) to \( E + p_0 \) preserving the condition \( C(g) \).

3. The main result. We now state and prove the main result of this paper.

**Theorem.** Let \( M \) be a separable metric space, let \( g(t) \in \mathcal{P}(M) \) and let \( x = \phi(p) \) be a transformation from a set \( E \) in \( M \) into a Euclidean space \( E_n \). Then if \( \phi \) satisfies the condition \( C(g) \) on \( E \), \( \phi \) can be extended to \( M \) preserving the condition \( C(g) \).

**Proof.** Let \( D \) be a finite or denumerable set which is dense in \( M \). By Lemma 2.3, \( \phi(p) \) can be extended to \( E \) plus any point of \( D \) and by induction to \( E + D \) preserving the condition \( C(g) \). Let \( x = \Phi(p), \ p \in E + D \) be the extended transformation. Since \( E_n \) is complete and \( g(t) \in \mathcal{P}(M) \), a convergent sequence of points \( p_m \in E + D, \ m = 1, 2, \ldots \), determines a convergent sequence of points \( \Phi(p_m) \) in \( E_n \). Since \( E + D \) is dense in \( M \), \( \Phi(p) \) can be extended to \( M \) preserving the condition \( C(g) \) in one and only one way.

4. Additional remarks. The writer is indebted to the referee for pointing out the following facts. Any finite set of points in a unitary space is isometrically equivalent to a set of points in some Euclidean space. Hence Lemma 2.2 is valid in any unitary space. Lemma 1.2 is valid in any complete unitary space (see Murray\(^6\) for the case where the space is separable and Alaoglu\(^7\) for the general case). Hence Lemma 2.3 is valid if \( E_n \) is replaced by a complete unitary space. Then the theorem in §3 with \( E_n \) replaced by a complete unitary space \( U \) and with \( M \) not assumed to be separable follows from Lemma 2.3 (with \( E_n \) replaced by \( U \)) by applying Zorn's lemma or transfinite induction.

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