ORTHOGONALITY PROPERTIES OF C-FRACTIONS

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1. Introduction. It has been indicated in the work of Tchebichef and Stieltjes that the denominators $D_p(z)$ of the successive approximants of a $J$-fraction

$$
\frac{b_0}{d_1 + z} - \frac{b_1}{d_2 + z} - \frac{b_2}{d_3 + z} - \cdots
$$

constitute a sequence of orthogonal polynomials. The orthogonality relations which exist between the $D_p(z)$ may be expressed in the following way (cf. [4, 7]). Let $S'$ be defined as the operator which replaces every $z^p$ by $c_p$ in any polynomial upon which it operates, where the \{c_p\} are a given sequence of constants. Then the orthogonality relations

$$
S'(D_p(z)D_q(z)) = \begin{cases} 0 & \text{for } p \neq q, \\ \neq 0 & \text{for } p = q,
\end{cases}
$$

hold relative to the operator $S'$ and the sequence \{c_p\}. The polynomials $D_p(z)$ are given recurrently by the formulas $D_0(z) = 1$, $D_p(z) = (d_p + z)D_{p-1}(z) - b_{p-1}D_{p-2}(z)$, $p = 1, 2, \cdots (D_{-1}(z) = 0)$.

In this paper orthogonality relations similar to (1.2) are developed for the polynomials $B_p^*(z)$ which are derived from the denominators $B_p(z)$ of the successive approximants of a $C$-fraction

$$
1 + \frac{a_1z^{\alpha_1}}{1 + \frac{a_2z^{\alpha_2}}{1 + \frac{a_3z^{\alpha_3}}{1 + \cdots}}}
$$

where the $a_p$ denote complex numbers and the $\alpha_p$ positive integers (cf. [3]). In fact, conditions (1.2) for a certain $J$-fraction are shown to be a specialization of the orthogonality relations for a $C$-fraction. Furthermore, necessary and sufficient conditions are obtained for the unique existence of the polynomials $B_p^*(z)$ in terms of the sequence \{c_p\} (Theorem 2.2).

2. Orthogonal polynomials constructed from the denominators of the approximants of a $C$-fraction. Let $A_p(z)$ and $B_p(z)$ denote the numerator and denominator, respectively, of the $p$th approximant

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1 Numbers in brackets refer to the bibliography at the end of the paper.
of a C-fraction (1.3). The recurrence formulas

$$
\begin{align*}
A_0(z) &= 1, & A_1(z) &= 1 + a_1 z, \\
A_{p+1}(z) &= A_p(z) + a_{p+1} z A_{p-1}(z), \\
B_0(z) &= 1, & B_1(z) &= 1, \\
B_{p+1}(z) &= B_p(z) + a_{p+1} z B_{p-1}(z),
\end{align*}
$$

(2.1)

show that $A_p(z)$ and $B_p(z)$ are polynomials of the form

$$
\begin{align*}
A_p(z) &= \gamma_0^{(p)} + \gamma_1^{(p)} z + \cdots + \gamma_p^{(p)} z^p, \\
B_p(z) &= 1 + \beta_1^{(p)} z + \cdots + \beta_p^{(p)} z^p.
\end{align*}
$$

(2.2)

From the determinant formula

$$
A_p(z) B_{p-1}(z) - A_{p-1}(z) B_p(z) = (-1)^p a_1 a_2 \cdots a_{p+1} z^{p+1},
$$

it follows that there is determined uniquely a power series

$$
P(z) = 1 + c_1 z + c_2 z^2 + \cdots
$$

(2.3)

such that the power series expansion for $A_p(z)/B_p(z)$ agrees term by term with the power series $P(z)$ up to the term involving $z^{p+1}$, that is,

$$
P(z) B_p(z) - A_p(z) = (-1)^p a_1 a_2 \cdots a_{p+1} z^{p+1}.
$$

This uniquely determined power series is called the corresponding power series.

In [3] it was shown that the algorithm

$$
\begin{pmatrix}
1 \\
\beta_1^{(p)} \\
\beta_2^{(p)} \\
\vdots \\
\vdots
\end{pmatrix}
$$

(2.4) \((c_n, c_{n-1}, c_{n-2}, \cdots)\)

combined with formulas (2.1), gives the $a_p$ and $\alpha_p$ of (1.3) corresponding to the power series $P(z)$ (2.3). Conversely, the coefficients $c_p$ of the power series expansion $P(z)$ are determined by (2.1) and (2.4) when the corresponding C-fraction (1.3) is given.

Let the polynomials $B_p^*(z)$ be defined as follows:
\[ B_p^*(z) = z^n B_p(1/z) = z^n + \beta_1^{(p)} z^{n-1} + \beta_2^{(p)} z^{n-2} + \cdots, \]
(2.5)
where \( n_p = \begin{cases} s_p & \text{if } s_p \geq t_p, \\ t_p & \text{if } s_p < t_p, \end{cases} \)

Let \( S \) be defined as the operator which replaces every \( z^p \) by \( c_{p+1} \) in any polynomial upon which it operates, and let the sequence
(2.6) \[ \{c_p\} : c_1, c_2, c_3, \ldots, \quad (c_0 = 1, c_i = 0, i < 0), \]
be the coefficients of the power series (2.3) corresponding to the \( C \)-fraction (1.3). From the algorithm (2.4)

(i) \[ S(z^p B_p^*(z)) = 0 \quad \text{for} \quad k = 0, 1, \ldots, \sum_{i=1}^{p+1} \alpha_i - n_p - 2, \]
(2.7)
(ii) \[ S(z^p B_p^*(z)) = (-1)^p a_1 a_2 \cdots a_{p+1} \quad \text{for} \quad k = \sum_{i=1}^{p+1} \alpha_i - n_p - 1, \]
\( p = 0, 1, \ldots. \)

Let \( B_p^*(z) = z^n + \beta_1^{(p)} z^{n-1} + \beta_2^{(p)} z^{n-2} + \cdots \) be one of the polynomials defined by (2.5). From (2.7) the following orthogonality relations hold between \( B_p^*(z) \) and \( B_p^*(z) \) relative to the operator \( S \) and the sequence \( \{c_p\} \):

(2.8) \[ S(B_p^*(z) B_q^*(z)) = \begin{cases} 0 & \text{for} \quad n_q = 0, 1, \ldots, \sum_{i=1}^{p+1} \alpha_i - n_p - 2, \\ (-1)^p a_1 a_2 \cdots a_{p+1} \neq 0 & \text{for} \quad n_q = \sum_{i=1}^{p+1} \alpha_i - n_p - 1, \quad p = 0, 1, \ldots. \end{cases} \]

Consequently the following theorem holds.

**Theorem 2.1.** Let the \( C \)-fraction (1.3) be given. By the algorithm (2.4) the sequence \( \{c_p\} \) (2.6) of coefficients of the corresponding power series (2.3) can be computed. Then the polynomials \( B_p^*(z) \), which are found by (2.5) from the denominators \( B_p(z) \) of the successive approximants of (1.3), satisfy the orthogonality relations (2.8) relative to the operator \( S \) and the sequence \( \{c_p\} \).

From formulas (2.4) or conditions (2.7) there follows directly a condition necessary for the existence of the polynomials \( B_p(z) \) and consequently the polynomials \( B_p^*(z) \), namely, that
\[ \Delta(0, 0) \neq 0, \Delta(0, i) = 0, i = 1, 2, \ldots, s_1 - 1, \Delta(0, s_1) \neq 0; \]
\[ \Delta(t_p - 1, s_p) \neq 0, \Delta(t_p - 1 + i, s_p + i) = 0, \]
\[ (2.9) \]
\[ \Delta \left( \sum_{i=1}^{p+1} \alpha_i - t_p - s_p - 1, \right) \neq 0, \quad p = 1, 2, \ldots, \]

where
\[ \Delta(j, k) = \begin{vmatrix} c_{k-j}, & c_{k-j+1}, & \cdots, & c_k \\ c_{k-j+1}, & c_{k-j+2}, & \cdots, & c_{k+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_k, & c_{k+1}, & \cdots, & c_{k+j} \end{vmatrix}, \quad j, k = 0, 1, \ldots \]

(cf. [3, (3.2)]). In fact, the coefficients \( a_p \neq 0 \) of (1.3) may be found in terms of the determinants \( \Delta \) provided conditions (2.9) hold.

Conversely, given a sequence \( \{c_p\} \) for which conditions (2.9) hold, one can construct a unique system of polynomials \( B^*_p(z) \) such that (2.7) hold, and consequently (2.8), and the \( B^*_p(z) \) are given by formulas (2.1), (2.5), and (2.7) (ii). For, by conditions (2.9), \( \Delta(0, 0) = c_0 = 1, \Delta(0, i) = 0, i = 1, \ldots, s_1 - 1, \) or \( c_1 = \cdots = c_{s_1-1} = 0, \) and \( \Delta(0, s_1) = c_{s_1} \neq 0, \) it follows that \( s_1 = c_1, \) since \( c_{s_1} \neq 0. \) Then equations (2.7) (i) hold, that is, \( S(z^k B^*_0(z)) = 0, \quad k = 0, \ldots, \alpha_1 - 2; \) if \( B^*_0(z) = B_0(z) = 1, \) and \( S(z^{\alpha_1-1} B^*_0(z)) = c_{\alpha_1} = a_1. \) Thus \( \alpha_1 \) and \( a_1 \) are determined. Suppose now \( B^*_1(z) = z^{\alpha_1} + \beta_1^{(1)} z^{\alpha_1-1} + \beta_2^{(1)} z^{\alpha_1-2} + \cdots = z^{\alpha_1} \) is computed by (2.1) and (2.5). By the algorithm (2.4)

\[ \begin{align*}
\sum_{i=0}^{\alpha_1} c_{\alpha_1-i+1} \beta_i^{(1)} &= 0, \quad j = 1, \ldots, \alpha_2 - 1, \\
\sum_{i=0}^{\alpha_1} c_{\alpha_1+\alpha_2-i} \beta_i^{(1)} &= -a_1 a_2.
\end{align*} \]

These equations are exactly conditions (2.7) for \( n_1 = \alpha_1. \) But since equations (2.7)(i) may be solved for the coefficients \( \beta^{(1)} \) uniquely because (2.9) hold, the polynomial \( B^*_1(z) \) as determined by (2.1) and (2.5) is the unique polynomial which satisfies (2.7). The value of \( a_2 \) may be determined from (2.11) and the value of \( \alpha_2 \) from (2.9). In exactly the same way one may show for \( p = 2, 3, \ldots \) that, from a given sequence \( \{c_p\} \) for which conditions (2.9) hold, unique polynomials \( B^*_p(z) \) may be found by (2.1), (2.5), and (2.7)(ii) such that (2.7) hold and consequently conditions (2.8). This completes the proof of the following theorem.
THEOREM 2.2. Let \( \{c_p\} \) be a given sequence of constants. There exists a unique sequence of polynomials \( B_p^*(z) \) (2.5) which satisfy conditions (2.7) and consequently relations (2.8) relative to the operator \( S \) and the sequence \( \{c_p\} \) if and only if conditions (2.9) hold. These polynomials are determined by the recurrence formulas (2.1), (2.5), and (2.7) (ii).

The polynomials \( B_p(z) \) (2.1) are the denominators of the successive approximants of the C-fraction (1.3) corresponding to the power series (2.3) with coefficients \( c_p \).

If conditions (2.9) hold for \( p = 1, \ldots, m \), then a finite sequence of orthogonal polynomials \( B_p^*(z) \) may be constructed. On the other hand, if (2.9) hold for all values of \( m \), there exists an infinite sequence of orthogonal polynomials \( B_p^*(z) \).

3. Special C-fractions. The C-fraction (1.3) (and its corresponding power series) is called regular if all of its approximants are Padé approximants (cf. [5]). In [3] necessary and sufficient conditions for regularity are found in terms of the \( s_p \) and \( t_p \) (cf. (2.2)). There is a special class of regular C-fractions called \( \alpha \)-regular, in which the condition of regularity depends only upon the exponents \( \alpha_p \), that is, the \( \alpha_p \) must satisfy the relations

\[
\begin{align*}
\alpha_1 + \alpha_3 + \cdots + \alpha_{2p+1} &= s_{2p+1} \geq \omega + 1 + t_{2p+1}, \\
\alpha_0 + \alpha_2 + \cdots + \alpha_{2p} &= t_{2p} \geq s_{2p} - \omega, \quad p = 0, 1, \ldots,
\end{align*}
\]

where \( \omega \) is an integer. In [3] it is shown that necessary and sufficient for \( \alpha \)-regularity are the conditions

\[
\Delta(i, \omega + 1 + i) = 0, \quad i = 0, 1, \ldots, h_1 - \omega - 2,
\]

\[
\Delta(h_1 - \omega - 1, h_1) \neq 0;
\]

\[
\Delta(h_p - \omega - 2, h_p) \neq 0, \Delta(h_p - \omega - 1 + i, h_p + 1 + i) = 0
\]

\[
i = 0, 1, \ldots, g_p - h_p + \omega - 1,
\]

\[
\Delta(h_{p+1} - \omega - 1, h_{p+1}) \neq 0, \quad p = 1, 2, \ldots,
\]

where \( g_0 = 0, g_p = \alpha_2 + \alpha_4 + \cdots + \alpha_{2p}, h_p = \alpha_1 + \alpha_3 + \cdots + \alpha_{2p-1} \). Conditions (3.2) reduce to (2.9) when \( s_p \) and \( t_p \) satisfy (3.1).

For \( \alpha \)-regular continued fractions, let the polynomials \( B_p^*(z) \) be defined as follows:


Then the following theorem is a corollary to Theorem 2.2.

**Theorem 3.1.** Let \{c_p\} be a given sequence of constants which are the coefficients of an \(\alpha\)-regular power series. There exists a set of unique polynomials \(B^*_p(z)\) (3.3) which satisfy the orthogonality relations

\[
S(B^*_{2p+1}(z)B^*_q(z)) = \begin{cases} 
0 & \text{for } n_q < g_{p+1} - 1, \\
(-1)^{2p+1}a_1a_2 \cdots a_{2p+2} \neq 0 & \text{for } n_q = g_{p+1} - 1; \\
0 & \text{for } n_q < h_{p+1} - \omega - 1, \\
(-1)^{2p}a_1a_2 \cdots a_{2p+1} \neq 0 & \text{for } n_q = h_{p+1} - \omega - 1, \\
\end{cases}
\]

relative to the operator \(S\) and the sequence \(\{c_p\}\). These polynomials are determined by the recurrence formulas (2.1), (3.3), and the relations

\[
S(z^iB^*_{2p+1}(z)) = (-1)^{2p+1}a_1a_2 \cdots a_{2p+2} \quad \text{for } i = g_{p+1} - 1, \\
S(z^iB^*_p(z)) = (-1)^{2p}a_1a_2 \cdots a_{2p+1} \\
\quad \text{for } i = h_{p+1} - \omega - 1, \quad p = 0, 1, \ldots .
\]

The polynomials \(B_p(z)\) (2.1) are the denominators of the successive approximants of an \(\alpha\)-regular C-fraction.

It may be remarked that for an \(\alpha\)-regular C-fraction the orthogonality relations (3.4) depend entirely on the exponents \(\alpha_p\) of the C-fraction and the integer \(\omega\), and are independent of the coefficients \(a_p\).

A second special case of the orthogonality relations (2.8) is obtained from the denominators of the approximants of a certain continued fraction (1.1). If one specializes the C-fraction (1.3) by putting \(\alpha_i=1\), \(i=1, 2, \ldots\), and then replacing \(z\) by \(1/z\), the even part (cf. [5, p. 201]) of the resulting continued fraction after certain equivalence transformations is

\[
1 + \frac{a_1}{a_2 + z} - \frac{a_2a_3}{(a_3 + a_4) + z} - \frac{a_4a_5}{(a_5 + a_6) + z} - \cdots .
\]

This is a \(J\)-fraction for which the orthogonality relations (2.8) reduce to the known conditions (1.2).
4. Analogue to the Christoffel-Darboux formula. The denominators \( D_p \) of the successive approximants of a \( J \)-fraction (1.1) are connected by the identity (cf. [1], also [2])

\[
D_0(z)D_0(w) + D_1(z)D_1(w) + \cdots + D_p(z)D_p(w)
= \frac{D_{p+1}(z)D_p(w) - D_p(z)D_{p+1}(w)}{z - w}, \quad p = 0, 1, \ldots.
\]

A similar relation for the denominators of the approximants of a \( C \)-fraction is shown in the following theorem.

**Theorem 4.1.** The denominators \( B_p \) of the approximants of a \( C \)-fraction (1.3) are connected by the identity

\[
B_{p+1}(z)B_p(w) - B_p(z)B_{p+1}(w) = \sigma_p \left[ a_{p+1} B_{p+1}(z) - a_p B_p(w) \right]
+ \sigma_{p+1} \left[ a_{p+1} B_{p+1}(w) - a_p B_p(z) \right]
+ \cdots + a_{p+1} a_p \cdots a_2 B_0(z)B_0(w) \sigma_{p+1} \sigma_p \cdots
- \sigma_{p+1} \sigma_p \cdots
\]

**Proof.** (4.1) is derived from the recurrence formula (2.1) for \( B_p \). For \( B_{p+1}(z)B_p(w) - B_p(z)B_{p+1}(w) = B_p(w) \left[ B_p(z) + a_{p+1} \sigma_p B_{p-1}(z) \right] - B_p(z) \left[ B_p(w) + a_{p+1} \sigma_p B_{p-1}(w) \right] = a_{p+1} \sigma_{p+1} B_{p+1}(w) \sigma_p B_{p-1}(z) + a_{p+1} a_p \cdots a_2 B_0(z)B_0(w) \sigma_{p+1} \sigma_p \sigma_{p-1} \cdots \]

By a repetition of this process, formula (4.1) is ultimately obtained.

**Bibliography**


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