REMARKS ON THE NOTION OF RECURRENCE

J. WOLFOWITZ

We give in several lines a simple proof of Poincaré's recurrence theorem.

THEOREM. Let \( \Omega \) be a point set of finite Lebesgue measure, and \( T \) a one-to-one measure-preserving transformation of \( \Omega \) into itself.\(^1\) Let \( B \subset A \subset \Omega \) be measurable sets such that, if \( b \in B \), \( T^m b \in A \) for all positive integral \( m \). Then the measure \( m(B) \) of \( B \) is 0.

PROOF. First we show that, if \( i < j \), \( (T^i B)(T^j B) = 0 \). Suppose \( c \in T^i B \); then from the hypothesis on \( B \) it follows that \( j \) is the smallest integer such that \( T^{-i} c \in A \). Hence \( c \in T^j B \). Now if \( m(B) = \delta > 0 \), \( \Omega \) would contain infinitely many disjunct sets \( T^k B \), each of measure \( \delta \). This contradiction proves the theorem.

The following generalization of the above theorem is trivially obvious: The result holds if we replace the hypothesis that \( T \) is measure-preserving by the following: If \( m(D) > 0 \), \( \lim \sup_i m\{T^i(D)\} > 0 \).

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Another obvious generalization is this: Let $C$ be the set of all points $c$ of $A$ such that $T^n c \in A$ for only finitely many $n$. Then $m(C) = 0$ (for $C \subset \sum_{i=0}^{\infty} T^{-i}B$).

The following is a simple derivation of Kac’s theorem on the mean recurrence time.²

**Theorem.** Let $T$ above be metrically transitive. Let $a \in A - B$, and $n(a)$ be the smallest positive integer such that $T^n a \in A$. Let $m(A) > 0$. Then

$$\int_{A - B} n(a) \, dm = m(\Omega).$$

**Proof.** Define $A_k = \{ n(a) = k \}$. Let $i < j$, $i' < j'$, $j \neq j'$. We notice:

(a) $(T^iA_j)(T^{i'}A_{j'}) = 0$. For $T$ has a single-valued inverse and $A_iA_{j'} = 0$. If $T^iA_j$ and $T^{i'}A_{j'}$ had a point $s$ in common, then $T^{-i}s \in A_j$, $T^{-i'}s \in A_{j'}$, in violation of the definition of $j$ and $j'$.

(b) $$\int_{A - B} n(a) \, dm = m \left( \sum_{k=1}^{\infty} \sum_{l=0}^{k-1} T^lA_h \right).$$

(c) Metric transitivity implies that almost every point in $\Omega$ lies in some $T^iA_h$, that is, $m(\sum \sum T^iA_h) = m(\Omega)$.

This proves the desired result.

² Kac, loc. cit. Theorem 2.