also *Selecta*, Paris, 1939, pp. 203–233.

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6. H. Hopf und H. Samelson, *Ein Satz über die Wirkungsräume geschlossener Liescher Gruppen*, Comment Math. Helv. vol. 13 (1941), pp. 240–251.

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ZÜRICH, SWITZERLAND

A NOTE ON LEAST COMMON LEFT MULTIPLES

B. M. STEWART

1. Introduction. Consider n-by-n matrices $A, B, \cdots$ with elements in a principal ideal ring and recall the following definitions. If $A=BC$, then A is a *left multiple* of C and C is a *right divisor* of A. If $A=RD$ and $B=PD$, then D is a *common right divisor* of A and B; if, furthermore, D is a left multiple of every common right divisor of A and B, then D is a *greatest common right divisor* of A and B. If $M=PA=QB$, then M is a *common left multiple* of A and B; if, furthermore, M is a right divisor of every common left multiple of A and B, then M is a *least common left multiple* of A and B. If $FE=I$, where I is the identity matrix, then E is a *unimodular* matrix. If E is unimodular, then EA is a *left associate* of A.

The basic tool in the following constructions is the theorem[1] that any given matrix A is the left associate of a uniquely determined matrix A_1, known as the Hermite canonical triangular form, having zeros above the main diagonal, having elements below the main diagonal in a prescribed residue class modulo the diagonal element above, having each diagonal element in a prescribed system of non-associates, and if a diagonal element is zero, having the corresponding row all zero.

C. C. MacDuffee has presented the following method,[2] due in essence to E. Cahen and A. Chatelet, for finding a greatest common

Presented to the Society, September 7, 1948; received by the editors April 30, 1948.

[1] C. C. MacDuffee, *Matrices with elements in a principal ideal ring*, Bull. Amer. Math. Soc. vol. 39 (1933) pp. 570–573.

[2] C. C. MacDuffee, loc. cit. p. 573.

right divisor D and a least common left multiple M of two given matrices A and B. Consider $2n$-by-$2n$ matrices written in the form of n-by-n blocks. Use the theory of the Hermite form, as applied to the $2n$-by-$2n$ matrices, to write the following equations.

$$\begin{pmatrix} R & S \\ P & Q \end{pmatrix}\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} R' & S' \\ P' & Q' \end{pmatrix}\begin{pmatrix} R & S \\ P & Q \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{1}$$
$$\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} R' & S' \\ P' & Q' \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}.$$

Since first: $A = R'D$ and $B = P'D$, and second: $RA + SB = D$, it follows that D is a greatest common right divisor of A and B, with no restrictions on A and B.

With the rather severe restriction that both A and B be nonsingular, it has been shown[3] that the matrix $M = PA = -QB$, obtained from (1), is a least common left multiple of A and B.

It is the purpose of this note to show by a different proof that the stated method for finding M is correct for a much larger set of matrices A and B—namely, whenever D is nonsingular, that is, whenever the row space of the union of A and B has the rank n, thus including, in addition to the case that both A and B are nonsingular, the case when either A or B is nonsingular and many other possibilities. Furthermore, in the exceptional case when D is singular, there is presented a method for finding an M that may be considered a natural modification of the method based on the equations (1). In conclusion it is noted that a theorem of E. Steinitz guarantees that both D and M are unique up to a unimodular left factor.

For completeness it should be noted that a method for finding the least common left multiple of two matrices when the matrices have elements in a field has been given by C. C. MacDuffee,[4] but the method is not applicable to the case where the elements are limited to a principal ideal ring.

2. **When D is nonsingular.** In terms of the notation explained in the introduction the following theorem may be stated:

THEOREM 1. *In the matric equation*

$$\begin{pmatrix} R & S \\ P & Q \end{pmatrix}\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

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[3] C. C. MacDuffee, loc. cit. p. 574.

[4] C. C. MacDuffee, *Vectors and matrices*, Carus Monograph, 1943, pp. 100–102.

where the matrix involving R, S, P, Q is unimodular, the matrix D is an all cases a greatest common right divisor of A and B; and if D is nonsingular, then the matrix $M=PA=-QB$ is a least common left multiple of A and B.

That D is in all cases a greatest common right divisor has been shown above. To show that M, which is obviously a common left multiple of A and B, is a least common left multiple, let $M_1=UA=-VB$ be any common left multiple of A and B. Use the matrices determined in (1) to write the following equations:

$$(2)\qquad \begin{aligned} \begin{pmatrix} R & S \\ U & V \end{pmatrix}\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} &= \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} R & S \\ U & V \end{pmatrix}\begin{pmatrix} R' & S' \\ P' & Q' \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Consider the most general solution of the equation

$$\begin{pmatrix} X & W \\ Y & T \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}.$$

Here W and T are arbitrary, but X and Y must be chosen so that

$$(3)\qquad XD = D, \qquad YD = 0.$$

Subject to the conditions (3) the following equations must hold:

$$\begin{pmatrix} R & S \\ U & V \end{pmatrix}\begin{pmatrix} R' & S' \\ P' & Q' \end{pmatrix} = \begin{pmatrix} X & W \\ Y & T \end{pmatrix}, \quad \begin{pmatrix} R & S \\ U & V \end{pmatrix} = \begin{pmatrix} X & W \\ Y & T \end{pmatrix}\begin{pmatrix} R & S \\ P & Q \end{pmatrix}.$$

In particular, it appears that U must have the form

$$(4)\qquad U = YR + TP.$$

But under the hypothesis that D is nonsingular the only solution of $YD=0$ is $Y=0$. Hence it follows in this case that $U=TP$; then from $M_1=UA=TPA=TM$ it follows that $M=PA=-QB$ is indeed a least common left multiple of A and B and the proof of the theorem is complete.

3. **When D is singular.** If the matrix D is singular it is easy to construct examples in which the matrix PA of Theorem 1 is not a least common left multiple of A and B. Hence the construction proposed in the following theorem has some interest.

THEOREM 2. *Let A_1 and B_1 be the Hermite forms of A and B, respectively. In the matric equation*

$$(1') \qquad \begin{pmatrix} R^* & S^* \\ P^* & Q^* \end{pmatrix}\begin{pmatrix} A_1 & 0 \\ B_1 & 0 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

where the matrix involving R^*, S^*, P^*, Q^* is unimodular and the matrix involving D is in Hermite form, let it be agreed, since this is a possible procedure, that no operations are performed upon the rows corresponding to the zero rows of D. Then the matrix $M^* = P^*A_1 = -Q^*B_1$ is a least common left multiple of A and B.

It follows readily that if E_1 and E_2 are unimodular, then a least common left multiple M^* of $A_1 = E_1A$ and $B_1 = E_2B$ is also a least common left multiple of A and B. First: from $M^* = KA_1 = LB_1$ it follows that $M^* = KE_1A = LE_2B$; second: from $M_1 = UA = VB$ it follows that $M_1 = UF_1A_1 = VF_2B_1$, where $F_1E_1 = I = F_2E_2$, but since M^* is a least common left multiple of A_1 and B_1, it follows that there exists a matrix G such that $M_1 = GM^*$. In particular A_1 and B_1 may be taken to be the Hermite forms of A and B, respectively.

In the equation (1′) the * notation has been adopted in describing the component blocks of the transforming matrix for two reasons: first: to indicate that the operations are to be made on the matrix involving A_1 and B_1 rather than A and B; and second: to indicate that the operations are to be made in what may be described as one of the "simplest" ways. Certainly no zero row can appear in D unless it is also a zero row of A_1 (the converse, of course, does not hold). Consequently, in obtaining the Hermite form involving D, it is quite unnecessary to operate on the rows (z) which are zero rows of D, and this agreement results in a "simplest" transforming matrix. In fact the matrix R^* must have each of its rows (z) all zero except for a 1 in the diagonal position. Furthermore, the most general matrix Y such that $YD = 0$ must have the form $Y = ZY^*$, where Z is arbitrary but Y^* has a structure just like R^* in its rows (z) and zero rows elsewhere. Thus $Y^*R^* = Y^*$ and

$$(5) \qquad YR^* = ZY^*R^* = ZY^* = Y.$$

As observed above A_1 has zero rows in at least the rows (z), hence from $YD = 0$ it follows that $YA_1 = 0$, and from (5) it follows that

$$(6) \qquad YR^*A_1 = YA_1 = 0.$$

Let $M_1 = UA_1 = -VB_1$ be any common left multiple of A_1 and B_1. Starting from (1′) instead of (1), obtain the relations analogous to (2), (3), and (4). In particular, consider

$$(4') \qquad U = YR^* + TP^*.$$

From (6) it follows that $M_1 = UA_1 = (YR^* + TP^*)A_1 = TP^*A_1$. Hence with $M^* = P^*A_1 = -Q^*B_1$ it follows that $M_1 = TM^*$ and that M^* is a least common left multiple of A_1 and B_1. Automatically M^* is a least common left multiple of A and B and the proof of the theorem is complete.

It should be noted that Theorem 2 works in all cases and does not *require* D to be singular. However the construction of Theorem 1, when applicable, is much shorter, since the separate computation of A_1 and B_1 is avoided. A working procedure would be to start from A and B and compute D in Hermite form; if D has no zero rows, apply Theorem 1; if D has zero rows (z), return to compute A_1 and B_1, find D anew starting from A_1 and B_1 but not operating on the rows (z), and then apply Theorem 2.

4. Uniqueness.

LEMMA. *If $A = UB$ and $B = VA$, then there exists a unimodular matrix E such that $B = EA$.*

This lemma has been proved by E. Steinitz[5] in a much more general setting—namely, for matrices whose elements are in a domain of integrity over the rational field, not necessarily a principal ideal ring. The proof is considerably simpler for the case of principal ideal rings. In fact by taking the proof for the case of matrices with elements in a field[6] and merely replacing the word nonsingular by the word unimodular, a suitable proof for the case of principal ideal rings is easily made.

THEOREM 3. *The greatest common right divisor D and the least common left multiple M of two given matrices A and B are uniquely determined up to unimodular left factors.*

If D and D_1 are two greatest common right divisors of A and B, then each is a common left multiple of the other, say $D = UD_1$ and $D_1 = VD$. Then by the lemma, D and D_1 are left associates.

If M and M_1 are two least common left multiples of A and B, then each is a common right divisor of the other, say $M_1 = UM$ and $M = VM_1$. Then by the lemma, M and M_1 are left associates.

MICHIGAN STATE COLLEGE

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[5] E. Steinitz, Math. Ann. vol. 71 (1911) pp. 328–354.

[6] C. C. MacDuffee, *Vectors and matrices*, pp. 45–46.