A BOUND FOR THE MEAN VALUE OF A FUNCTION

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Let $f(t)$ be a bounded measurable function defined when $0 \leq t \leq \pi$. The Fourier sine series associated with $f(t)$ is

$$
\sum_{n=1}^{\infty} b_n \sin nt, \quad b_n = \frac{2}{\pi} \int_{0}^{\pi} f(t) \sin nt \, dt.
$$

We shall be interested in this paper in establishing a bound for the mean value

$$
a = \frac{1}{\pi} \int_{0}^{\pi} f(t) \, dt
$$

when $f(t)$ is such that one of the coefficients $b_n$ vanishes.

We can suppose without essential loss of generality that $|f(t)| \leq 1$. Since $b_{2n}=0$ whenever $f(t)$ is constant, it is clear that the only conclusion on $a$ that can be drawn from the inequality $|f(t)| \leq 1$ and the equality $b_{2n}=0$ is that $|a| \leq 1$, and this conclusion is valid whether $b_{2n}$ vanishes or not. Hence we shall restrict attention to $b_{2n+1}$. For the same reason we shall not discuss the vanishing of the coefficient

$$
a_n = \frac{2}{\pi} \int_{0}^{\pi} f(t) \cos nt \, dt
$$

of the Fourier cosine series of $f(t)$.

Suppose that $b_{2n+1}=0$. Define a positive number $y$ by the equation

$$
y = \sin \left[ \sec^{-1}(2n+2) \right]
$$

where the sec$^{-1}$ lies between 0 and $\pi/2$. Let $E$ be the sum of the intervals

$$
\frac{2\phi \pi + \sin^{-1}y}{2n+1} \leq t \leq \frac{(2\phi + 1)\pi - \sin^{-1}y}{2n+1} \quad (\phi = 0, 1, \ldots, n),
$$

where the sin$^{-1}$ lies between 0 and $\pi/2$. Then it is clear that

$$
\begin{align*}
\sin (2n + 1)t &\geq y & \text{if } t \text{ is in } E, \\
\sin (2n + 1)t &< y & \text{if } t \text{ is not in } E.
\end{align*}
$$

Received by the editors February 11, 1948, and, in revised form, June 14, 1948.

1 The importance of the concept of mean value in the study of Fourier series can be seen by consulting Bohr [1, pp. 7–29]. Numbers in brackets refer to the bibliography at the end of the paper.
Now let \( f_0(t) = -1 \) if \( t \) is in \( E \) and \( f_0(t) = +1 \) if \( t \) is not in \( E \). It follows from the definitions of \( y \) and \( E \) that
\[
\int_0^\infty f_0(t) \sin (2n + 1)t \, dt = 0,
\]
and that the mean value of \( f_0(t) \) is
\[
a_0 = 1 - (2/\pi) \text{meas } E
\]
\[
= [4(n + 1)/(2n + 1)] \sec^{-1} (2n + 2) - 1/(2n + 1).
\]

We shall now prove that if \( f(t) \) is an arbitrary real-valued measurable function such that \( |f(t)| \leq 1 \) and \( b_{2n+1} = 0 \), then \( a \leq a_0 \). Let \( g(t) = f(t) - f_0(t) \). Then \( 0 \leq g(t) \) on \( E \) and \( 0 \geq g(t) \) on the complement \( cE \) of \( E \). By virtue of relations (1) we conclude that
\[
\int_E g(t) \sin (2n + 1)t \, dt \geq y \int_E g(t) \, dt,
\]
\[
\int_{cE} g(t) \sin (2n + 1)t \, dt \geq y \int_{cE} g(t) \, dt.
\]
Adding and remembering that \( b_{2n+1} = 0 \) for both \( f \) and \( f_0 \) we see that
\[
0 \geq y \int_0^\infty g(t) \, dt,
\]
with equality if and only if \( g(t) = 0 \) almost everywhere. Since \( y > 0 \), we have that
\[
(2) \quad \int_0^\infty f(t) \, dt \leq \int_0^\infty f_0(t) \, dt,
\]
with equality if and only if \( f(t) = f_0(t) \) almost everywhere.

Now let \( h(t) = f(t) + f_0(t) \). Then \( 0 \geq h(t) \) on \( E \), \( 0 \leq h(t) \) on \( cE \), and so
\[
\int_E h(t) \sin (2n + 1)t \, dt \leq y \int_E h(t) \, dt,
\]
\[
\int_{cE} h(t) \sin (2n + 1)t \, dt \leq y \int_{cE} h(t) \, dt,
\]
\[
0 \leq y \int_0^\infty h(t) \, dt,
\]
\[
(3) \quad - \int_0^\infty f(t) \, dt \leq \int_0^\infty f_0(t) \, dt,
\]
with equality if and only if \( f(t) = -f_0(t) \) almost everywhere. Combining the inequalities (2) and (3) we conclude that when \( f(t) \) is a real-valued measurable function such that \( |f(t)| \leq 1 \) and \( b_{2n+1} = 0 \), then

\[
|a| = \left| \frac{1}{\pi} \int_{0}^{\pi} f(t) dt \right| \leq \frac{4(n + 1)}{(2n + 1)\pi} \sec^{-1} (2n + 2) - \frac{1}{2n + 1},
\]

with equality if and only if \( f(t) = \pm f_0(t) \) almost everywhere.

In particular, if \( b_1 = 0 \), then \( |a| \leq 1/3 = .3333 \), while if \( b_2 = 0 \), then \( |a| \leq .7855 \). The right-hand side of the inequality (4) approaches unity as \( n \) approaches infinity.

This conclusion may be extended to complex functions \( f(t) \) as follows. Let \( f(t) = f_1(t) + if_2(t) \), where \( f_1(t) \) and \( f_2(t) \) are real. There exist real numbers \( x \) and \( y \) such that

\[
x^2 + y^2 = 1, \quad x \int_{0}^{\pi} f_2(t) dt + y \int_{0}^{\pi} f_1(t) dt = 0.
\]

Hence it is true that the mean value of \( f(t) \) has the same absolute value as the mean value of the real function \( xf_1(t) - yf_2(t) \). This real function has a Fourier coefficient \( b_{2n+1} \) equal to zero since this is true for both \( f_1(t) \) and \( f_2(t) \) and is bounded by one since \( f(t) \) is and \( x^2 + y^2 = 1 \). Since the inequality (4) is valid for \( xf_1 - yf_2 \), it is therefore true for \( f(t) \). Moreover since equality for \( xf_1 - yf_2 \) implies that \( xf_1 - yf_2 = \pm f_0(t) \), equality for \( f(t) \) implies that \( f(t) = cf_0(t) \) where \( c \) is a constant of absolute value unity.

**Bibliography**