REAL ROOTS OF DIRICHLET L-SERIES

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Let $k$ be a positive integer. Let $\chi$ be a real, non-principal character (mod $k$) and

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

be the corresponding $L$-series, which converges uniformly for $R(s) \geq \epsilon > 0$. If it could be shown that uniformly in $k$ there is no real zero of $L(s, \chi)$ for

$$s \geq 1 - \frac{A}{\log k},$$

where $A$ is a constant, then the existing theorems on the distribution of primes in arithmetic progressions could be greatly improved (see [1]). Moreover by Hecke's Theorem (see [2]) it would follow that uniformly in $k$

$$L(1, \chi) > \frac{B}{\log k}$$

where $B$ is a constant. This would be a considerable improvement over Siegel's Theorem (see [3]), and would lead to an improved lower bound for the class number of an imaginary quadratic field.

In the present paper, we shall show that for $2 \leq k \leq 67$, $L(s, \chi)$ has no positive real zeros. By combining this information with the results of [1], we infer very sharp estimates on the distribution of primes in arithmetic progressions of difference $k$ for $k \leq 67$.

The methods used for $k \leq 67$ certainly will suffice for many other $k$'s greater than 67. They may possibly suffice for all $k$, but we can find no proof of this.\(^2\)

In [5], S. Chowla has considered the positive real zeros of $L(s, \chi)$, and shown that for many explicit $k$'s, no positive real zeros exist. However Chowla could not settle whether his methods would suffice

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\(^1\) Numbers in brackets refer to the bibliography at the end of the paper.

\(^2\) These methods have been tried on all $k \leq 227$ and it has been ascertained that except for the cases $k = 148$ and $k = 163$, $L(s, \chi)$ has no positive real zeros for $2 \leq k \leq 227$. Cases $k = 148$ and $k = 163$ are now being studied and any results obtained about them will appear in the Journal of Research of the National Bureau of Standards.

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to handle the difficult cases \( k = 43 \) and \( k = 67 \). In [6], Heilbronn has shown that there exist values of \( k \) for which Chowla's methods are certainly inadequate.

**Theorem 1.** If \( \chi \) is non-principal \((\text{mod } k)\) and \( \chi(-1) = 1 \), then for all \( s \)

\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{2s(s+1) \cdots (s+2\alpha-1)}{4^\alpha(2\alpha)!k^{n+2\alpha}} (2s+2\alpha-1)\xi(s+2\alpha) \\
\cdot \sum_{n=1}^{\lfloor k/2 \rfloor} \chi(n)(k-2n)^{2\alpha}.
\]

**Proof.** For \( s > 1 \), we have

\[
L(s, \chi) = 2^s \sum_{N=0}^{\infty} \sum_{n=1}^{k-1} \frac{\chi(n)}{(2kN+2n)^s} \\
= 2^s \sum_{N=0}^{\infty} \frac{1}{k^s(2N+1)^s} \sum_{n=1}^{k-1} \chi(n) \left(1 - \frac{k-2n}{k(2N+1)}\right)^{-s} \\
= 2^s \sum_{N=0}^{\infty} \frac{1}{k^s(2N+1)^s} \sum_{n=1}^{k-1} \chi(n) \left\{1 + s \frac{k-2n}{k(2N+1)} + \cdots \right\} \\
= 2^s \sum_{N=0}^{\infty} \frac{1}{k^s(2N+1)^s} \left\{\frac{s(s+1)}{3!} \left(\frac{k-2n}{k(2N+1)}\right)^3 + \cdots \right\} \\
= \frac{s}{2k^{s+1}} \left(\sum_{N=0}^{\infty} \left(\frac{2}{2N+1}\right)^{s+1}\right) \sum_{n=1}^{k-1} \chi(n)(k-2n) \\
+ \frac{s(s+1)}{4(2!)k^{s+2}} \left(\sum_{N=0}^{\infty} \left(\frac{2}{2N+1}\right)^{s+2}\right) \sum_{n=1}^{k-1} \chi(n)(k-2n)^2 + \cdots \\
= \frac{s}{2k^{s+1}} (2^{s+1} - 1)\xi(s+1) \sum_{n=1}^{k-1} \chi(n)(k-2n) \\
+ \frac{s(s+1)}{4(2!)k^{s+2}} (2^{s+2} - 1)\xi(s+2) \sum_{n=1}^{k-1} \chi(n)(k-2n)^2 + \cdots .
\]
Since \( \chi \) is non-principal, we have \( k > 2 \), and so if \( k \) is even, we have
\[
\chi([k/2]) = \chi(k/2) = 0.
\]
Now since \( \chi(-1) = 1 \),
\[
\sum_{n=1}^{[k/2]} \chi(n)(k - 2n)^{2\alpha}
= \sum_{n=1}^{[k/2]} \chi(n)(k - 2n)^{2\alpha} + \sum_{n=[k/2]+1}^{k-1} \chi(n)(2n - k)^{2\alpha}
= \sum_{n=1}^{[k/2]} \chi(n)(k - 2n)^{2\alpha} + \sum_{n=k-[k/2]}^{[k/2]} \chi(n)(2(n - k))^{2\alpha}
= \sum_{n=1}^{[k/2]} \chi(n)(k - 2n)^{2\alpha} + \sum_{n=1}^{[k/2]} \chi(k - n)(k - 2n)^{2\alpha}
= 2 \sum_{n=1}^{[k/2]} \chi(n)(k - 2n)^{2\alpha}.
\]
Similarly, we prove
\[
\sum_{n=1}^{k-1} \chi(n)(k - 2n)^{2\alpha+1} = 0.
\]
Thus we infer that the equation stated is valid for \( s > 1 \).

Now since
\[
\left| \sum_{n=1}^{[k/2]} \chi(n)(k - 2n)^{2\alpha} \right| \leq \frac{k}{2} (k - 2)^{2\alpha},
\]
we see that the series on the right converges absolutely and uniformly for all \( s \), and so our theorem follows by analytic continuation.

**Theorem 2.** If \( \chi \) is non-principal \((\text{mod } k)\) and \( \chi(-1) = -1 \), then for all \( s \)
\[
L(s, \chi) = \sum_{\alpha=0}^{\infty} \frac{s(s + 1) \cdots (s + 2\alpha)}{4^\alpha (2\alpha + 1)! k^{s+2\alpha+1}} (2^{s+2\alpha+1} - 1) \zeta(s + 2\alpha + 1)
\cdot \sum_{n=1}^{[k/2]} \chi(n)(k - 2n)^{2\alpha+1}.
\]

The proof is similar to the proof of Theorem 1.

Although these theorems hold for any non-principal \( \chi \), we shall use them only for real non-principal \( \chi \). We assume henceforth that \( \chi \) is real and non-principal. We let \( \Sigma_M \) denote
\[
\sum_{n=1}^{[k/2]} \chi(n)(k - 2n)^M.
\]

For sufficiently large \( M \) (certainly for \( M \geq k \)), the initial term
\[
\chi(1)(k - 2)^M
\]
of \( \Sigma_M \) dominates the remaining terms, and we infer that \( \Sigma_M > 0 \). If by good chance \( \Sigma_M \geq 0 \) for all \( M \geq 1 \), then by Theorem 1 or Theorem 2 we infer that \( L(s, \chi) > 0 \) for \( s > 0 \), and hence that \( L(s, \chi) \) has no positive real zeros. For \( k \leq 67 \), this happens in a majority of cases.

When considering positive real zeros of \( L(s, \chi) \) it suffices to restrict attention to primitive \( \chi \)'s (and to the \( k \)'s for which there are primitive \( \chi \)'s. See [4, §125]). For primitive \( \chi \)'s, \( \Sigma_M \geq 0 \) for \( M \geq 1 \) for each \( k \leq 67 \) except 43 and 67. Moreover for each such \( k \), the proof of \( \Sigma_M \geq 0 \) is easily accomplished by grouping the terms in groups, each of which is non-negative. Typical such groups are:

I. \( A^M - B^M \), where \( A > B \).

II. \( A^M - B^M - C^M \), where \( A \geq B + C \).

III. \( A^M - B^M - C^M + D^M \), where \( A + D \geq B + C \).

For \( k = 53 \), there occurs the group \( 51^M - 49^M - 47^M + 45^M - 43^M + 41^M - 39^M - 37^M \), which we show to be non-negative by writing it as \( (44 + 7)^M - (44 + 5)^M - (44 + 3)^M - (44 + 1)^M - (44 - 1)^M + (44 - 3)^M + (44 - 5)^M - (44 - 7)^M \), and expanding each term by the binomial theorem.

For \( k = 43 \) or 67, we have \( \Sigma_2 < 0 \), so that the series in Theorem 2 does not consist entirely of non-negative terms. However, we can show that the initial positive term outweighs the negative terms. We give the proof for \( k = 67 \), since the proof for \( k = 43 \) is similar and easier.

By the functional equation for \( L(s, \chi) \) (see [4, §128]) it follows that if \( L(s, \chi) \) has a zero \( \rho \) with \( 1/2 < \rho < 1 \), then it has a zero \( \rho \) with \( 0 < \rho < 1/2 \). As it is known that \( L(s, \chi) > 0 \) for \( 1 \leq s \), it suffices to prove \( L(s, \chi) > 0 \) for \( 0 \leq s \leq 1/2 \). So we take \( k = 67 \) and \( 0 \leq s \leq 1/2 \). By Theorem 2,

\[
L(s, \chi) = \frac{2^{s+1} - 1}{67^s} \left\{ \frac{s\zeta(s + 1)}{67} \Sigma_1 \right. \\
+ \frac{s(s + 1)(s + 2)}{6^s} \frac{2^{s+3} - 1}{3!(67)^3} \frac{\zeta(s + 3) \Sigma_3 + \cdots}{4(2^{s+1} - 1)},
\]

where now \( \Sigma_M = \sum_{n=1}^{67} \chi(n)(67 - 2n)^M \). For \( s > 0 \),

\[
\zeta(s + 1) - \frac{1}{s} = \sum_{n=1}^{\infty} \frac{1}{n^{s+1}} - \int_{1}^{\infty} \frac{dx}{x^{s+1}} = \sum_{n=1}^{\infty} \left\{ \frac{1}{n^{s+1}} - \int_{n}^{n+1} \frac{dx}{x^{s+1}} \right\} > 0.
\]

So for \( s \geq 0 \), \( s\zeta(s + 1) \geq 1 \). Also \( \Sigma_1 = 67 \). So
For $0 \leq s$
\[
\frac{2s+2a+1 - 1}{4a(2s+1 - 1)} \leq \frac{2s}{2 - 2^{-s}} \quad \text{and} \quad \frac{d}{ds} \left( \frac{2s}{2 - 2^{-s}} \right) > 0.
\]
So for $0 \leq s \leq 1/2$
\[
\frac{2s+2a+1 - 1}{4a(2s+1 - 1)} \leq \frac{2(1/2)}{2 - 2^{-1/2}} < 0.77346.
\]
Also
\[
\frac{(s+1)(s+2)}{3!} \leq \frac{(3/2) \cdot (5/2)}{3!} = \frac{5}{8}.
\]
Since $\Sigma_3 = -102,845$, we infer
\[
\frac{s(s+1)(s+2)}{3!(67)^3} \geq -\frac{5}{8} \left( \frac{1}{(67)^3} \right) (0.77346)^3 (102,845)
\]
\[
\geq -\frac{5}{8} \left( \frac{0.77346}{300,763} \right) 102,845 < 0.199.
\]
Now for $M \geq 1$,
\[
\Sigma_M = \{ (57 + 8)^M - (57 + 6)^M - (57 + 4)^M + (57 + 2)^M - 57^M 
+ (57 - 2)^M - (57 - 4)^M - (57 - 6)^M + (57 - 8)^M \}
+ \{ (43 + 4)^M - (43 + 2)^M - 43^M - (43 - 2)^M + (43 - 4)^M \}
+ 37^M + 35^M + \ldots
\]
\[
> -57^M + \frac{M(M - 1)}{2!} 57^{M-2} \{ 2 \cdot 8^2 - 2 \cdot 6^2 - 2 \cdot 4^2 + 2 \cdot 2^2 \}
+ \frac{M(M - 1)(M - 2)(M - 3)}{4!} 57^{M-4} \{ 2 \cdot 8^2 - 2 \cdot 6^2 - 2 \cdot 4^2 + 2 \cdot 2^2 \}
- 2 \cdot 4^4 + 2 \cdot 2^4 + \ldots
\]
\[
- 43^M + \frac{M(M - 1)}{2!} 43^{M-2} \{ 2 \cdot 4^2 - 2 \cdot 2^2 \} + \ldots
\]
\[ \sum_{\alpha=2}^\infty \frac{s(s + 1) \cdots (s + 2\alpha)}{(2\alpha + 1)!((67)^{2\alpha+1})} \cdot \frac{2^{2\alpha+1} - 1}{4^\alpha(2^{\alpha+1} - 1)} \xi(s + 2\alpha + 1) \]

\[ > - \frac{63}{128} (0.77346)(1.03693) \sum_{\alpha=2}^\infty \left\{ \frac{(57)^{2\alpha+1}}{67} \cdot \frac{2929}{3249} \right\} \]

\[ + \left( \frac{43}{67} \cdot \frac{1609}{1849} \right) \]

\[ > - 0.638. \]
By (1), (2), and (3), for \( 0 \leq s \leq 1/2 \),

\[
L(s, \chi) \geq \frac{2^{s+1} - 1}{6^s} \{ 1.000 - 0.199 - 0.638 \} \geq \frac{0.163}{(67)^{1/2}} \geq 0.0199.
\]

So \( L(s, \chi) > 0 \) for \( 0 \leq s \).

When \( \chi(-1) = -1 \), Theorem 2 opens up further interesting possibilities. When \( s \to 0 \), the first term of the series is bounded away from zero, while the remaining terms approach zero. Thus one can always infer \( L(s, \chi) > 0 \) for \( 0 \leq s \leq \epsilon \), where \( \epsilon \) depends on \( k \). Even for \( \epsilon \) as small as \( A/\log k \), this would be a very worthwhile result, as remarked at the beginning of the paper.

For another possibility, let \( s = 0 \) and \(-2\) in Theorem 2, and evaluate \( L(0, \chi) \) and \( L(-2, \chi) \) by the functional equation. We infer the known result

\[
L(1, \chi) = \frac{\pi}{k^{3/2}} \Sigma_1
\]

and the result

\[
L(3, \chi) = \frac{\pi^3}{6k^{7/2}} \{ k^3 \Sigma_1 - \Sigma_2 \}.
\]

From these follow

\[
\Sigma_3 = k^{7/2} \left\{ \frac{L(1, \chi)}{\pi} - \frac{6L(3, \chi)}{\pi^3} \right\}.
\]

This gives

\[
\Sigma_3 \geq - k^{7/2} \frac{6L(3, \chi)}{\pi^3}.
\]

If one could prove independently any appreciably better result, one could derive a sensational inequality for \( L(1, \chi) \). For instance, if one could prove

\[
\Sigma_3 \geq - k^{7/2} \frac{4}{\pi^3} \geq - k^{7/2} \frac{5L(3, \chi)}{\pi^3},
\]

one could get by (6)

\[
L(1, \chi) > \frac{L(3, \chi)}{\pi^2}.
\]

Another possibility is that one can perhaps derive some connec-
tion between $\Sigma_1$ and $\Sigma_3$. For instance, if one could prove

$$\Sigma_3 \geq -k^2 \log k \Sigma_1,$$

then by (4) and (6), we could infer

$$L(1, \chi) > \frac{6L(3, \chi)}{\pi^2(1 + \log k)}.$$

Even this would be a very worthwhile result, since the best known at present is, by Siegel's Theorem,

$$L(1, \chi) > \frac{L(3, \chi)}{k^\epsilon}$$

for $\epsilon > 0$ and large $k$.

**Bibliography**


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