REAL ROOTS OF DIRICHLET L-SERIES

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Let \( k \) be a positive integer. Let \( \chi \) be a real, non-principal character \((\mod k)\) and

\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}
\]

be the corresponding \( L \)-series, which converges uniformly for \( \Re(s) \geq \epsilon > 0 \). If it could be shown that uniformly in \( k \) there is no real zero of \( L(s, \chi) \) for

\[
s \geq 1 - \frac{A}{\log k},
\]

where \( A \) is a constant, then the existing theorems on the distribution of primes in arithmetic progressions could be greatly improved (see [1]). Moreover by Hecke’s Theorem (see [2]) it would follow that uniformly in \( k \)

\[
L(1, \chi) > \frac{B}{\log k}
\]

where \( B \) is a constant. This would be a considerable improvement over Siegel’s Theorem (see [3]), and would lead to an improved lower bound for the class number of an imaginary quadratic field.

In the present paper, we shall show that for \( 2 \leq k \leq 67 \), \( L(s, \chi) \) has no positive real zeros. By combining this information with the results of [1], we infer very sharp estimates on the distribution of primes in arithmetic progressions of difference \( k \) for \( k \leq 67 \).

The methods used for \( k \leq 67 \) certainly will suffice for many other \( k \)'s greater than 67. They may possibly suffice for all \( k \), but we can find no proof of this.\(^2\)

In [5], S. Chowla has considered the positive real zeros of \( L(s, \chi) \), and shown that for many explicit \( k \)'s, no positive real zeros exist. However Chowla could not settle whether his methods would suffice

\(^1\) Numbers in brackets refer to the bibliography at the end of the paper.
\(^2\) These methods have been tried on all \( k \leq 227 \) and it has been ascertained that except for the cases \( k = 148 \) and \( k = 163 \), \( L(s, \chi) \) has no positive real zeros for \( 2 \leq k \leq 227 \). Cases \( k = 148 \) and \( k = 163 \) are now being studied and any results obtained about them will appear in the Journal of Research of the National Bureau of Standards.
to handle the difficult cases $k = 43$ and $k = 67$. In [6], Heilbronn has shown that there exist values of $k$ for which Chowla's methods are certainly inadequate.

**Theorem 1.** If $\chi$ is non-principal (mod $k$) and $\chi(-1) = 1$, then for all $s$

\[
L(s, \chi) = \sum_{\alpha=1}^{\infty} \frac{2s(s + 1) \cdots (s + 2\alpha - 1)}{4^\alpha (2\alpha)! k^{s+2\alpha}} (2s+2\alpha - 1) \zeta(s + 2\alpha) 
\cdot \sum_{n=1}^{[k/2]} \chi(n)(k - 2n)^{2\alpha}.
\]

**Proof.** For $s > 1$, we have

\[
L(s, \chi) = 2^s \sum_{N=0}^{\infty} \sum_{n=1}^{k-1} \frac{\chi(n)}{(2kN + 2n)^s}
\]

\[
= 2^s \sum_{N=0}^{\infty} \frac{1}{k^s (2N + 1)^s} \sum_{n=1}^{k-1} \chi(n) \left( 1 - \frac{k - 2n}{k(2N + 1)} \right)^{-s}
\]

\[
= 2^s \sum_{N=0}^{\infty} \frac{1}{k^s (2N + 1)^s} \sum_{n=1}^{k-1} \chi(n) \left( 1 + s \frac{k - 2n}{k(2N + 1)} 
\right)
\]

\[
+ \frac{s(s + 1)}{2!} \left( \frac{k - 2n}{k(2N + 1)} \right)^2 
\]

\[
+ \frac{s(s + 1)(s + 2)}{3!} \left( \frac{k - 2n}{k(2N + 1)} \right)^3 + \cdots \}
\]

\[
= 2^s \sum_{N=0}^{\infty} \frac{1}{k^s (2N + 1)^s} \left\{ \frac{s}{k(2N + 1)} \sum_{n=1}^{k-1} \chi(n)(k - 2n) 
\right.
\]

\[
+ \frac{s(s + 1)}{2! k^s (2N + 1)^s} \sum_{n=1}^{k-1} \chi(n)(k - 2n)^2 + \cdots \}\}
\]

\[
= \frac{s}{2 k^{s+1}} \left( \sum_{N=0}^{\infty} \left( \frac{2}{2N + 1} \right)^{s+1} \right) \sum_{n=1}^{k-1} \chi(n)(k - 2n)
\]

\[
+ \frac{s(s + 1)}{4(2!)} \frac{1}{k^{s+2}} \left( \sum_{N=0}^{\infty} \left( \frac{2}{2N + 1} \right)^{s+2} \right) \sum_{n=1}^{k-1} \chi(n)(k - 2n)^2 + \cdots 
\]

\[
= \frac{s}{2 k^{s+1}} (2^{s+1} - 1) \zeta(s + 1) \sum_{n=1}^{k-1} \chi(n)(k - 2n)
\]

\[
+ \frac{s(s + 1)}{4(2!)} \frac{1}{k^{s+2}} (2^{s+2} - 1) \zeta(s + 2) \sum_{n=1}^{k-1} \chi(n)(k - 2n)^2 + \cdots .
\]
Since \( \chi \) is non-principal, we have \( k > 2 \), and so if \( k \) is even, we have \( \chi([k/2]) = \chi(k/2) = 0 \). Now since \( \chi(-1) = 1 \),

\[
\sum_{n=1}^{k-1} \chi(n)(k - 2n)^{2a}
\]

\[
= \sum_{n=1}^{[k/2]} \chi(n)(k - 2n)^{2a} + \sum_{n=\lfloor k/2 \rfloor + 1}^{k-1} \chi(n)(2n - k)^{2a}
\]

\[
= \sum_{n=1}^{[k/2]} \chi(n)(k - 2n)^{2a} + \sum_{n=k-[k/2]}^{[k/2]} \chi(n)(2(k - n))^{2a}
\]

\[
= \sum_{n=1}^{[k/2]} \chi(n)(k - 2n)^{2a} + \sum_{n=1}^{[k/2]} \chi(k - n)(k - 2n)^{2a}
\]

\[
= 2 \sum_{n=1}^{[k/2]} \chi(n)(k - 2n)^{2a}.
\]

Similarly, we prove \( \sum_{n=1}^{k-1} \chi(n)(k - 2n)^{2a+1} = 0 \).

Thus we infer that the equation stated is valid for \( s > 1 \). Now since

\[
\left| \sum_{n=1}^{[k/2]} \chi(n)(k - 2n)^{2a} \right| \leq \frac{k}{2} (k - 2)^{2a},
\]

we see that the series on the right converges absolutely and uniformly for all \( s \), and so our theorem follows by analytic continuation.

**Theorem 2.** If \( \chi \) is non-principal (mod \( k \)) and \( \chi(-1) = -1 \), then for all \( s \)

\[
L(s, \chi) = \sum_{a=0}^{\infty} \frac{s(s + 1) \cdots (s + 2\alpha)}{4^\alpha(2\alpha + 1)!k^{\alpha + 2\alpha + 1}} \frac{(2s+2\alpha+1 - 1)\zeta(s + 2\alpha + 1)}{(2s+2\alpha+1)} \cdot \sum_{n=1}^{[k/2]} \chi(n)(k - 2n)^{2a+1}.
\]

The proof is similar to the proof of Theorem 1.

Although these theorems hold for any non-principal \( \chi \), we shall use them only for real non-principal \( \chi \). We assume henceforth that \( \chi \) is real and non-principal. We let \( \Sigma_M \) denote

\[
\sum_{n=1}^{[k/2]} \chi(n)(k - 2n)^M.
\]

For sufficiently large \( M \) (certainly for \( M \geq k \)), the initial term

\[
\chi(1)(k - 2)^M
\]
of $\Sigma_M$ dominates the remaining terms, and we infer that $\Sigma_M > 0$. If by good chance $\Sigma_M \geq 0$ for all $M \geq 1$, then by Theorem 1 or Theorem 2 we infer that $L(s, \chi) > 0$ for $s > 0$, and hence that $L(s, \chi)$ has no positive real zeros. For $k \leq 67$, this happens in a majority of cases.

When considering positive real zeros of $L(s, \chi)$ it suffices to restrict attention to primitive $\chi$'s (and to the $k$'s for which there are primitive $\chi$'s. See [4, §125]). For primitive $\chi$'s, $\Sigma_M \geq 0$ for $M \geq 1$ for each $k \leq 67$ except 43 and 67. Moreover for each such $k$, the proof of $\Sigma_M \geq 0$ is easily accomplished by grouping the terms in groups, each of which is non-negative. Typical such groups are:

I. $A^M - B^M$, where $A > B$.
II. $A^M - B^M - C^M$, where $A \geq B + C$.
III. $A^M - B^M - C^M + D^M$, where $A + D \geq B + C$.

For $k = 53$, there occurs the group $51^M - 49^M - 47^M + 45^M - 43^M + 41^M + 39^M - 37^M$, which we show to be non-negative by writing it as $(44 + 7)^M - (44 + 5)^M - (44 + 3)^M + (44 + 1)^M - (44 - 1)^M + (44 - 3)^M + (44 - 5)^M - (44 - 7)^M$, and expanding each term by the binomial theorem.

For $k = 43$ or 67, we have $\Sigma_3 < 0$, so that the series in Theorem 2 does not consist entirely of non-negative terms. However, we can show that the initial positive term outweighs the negative terms. We give the proof for $k = 67$, since the proof for $k = 43$ is similar and easier.

By the functional equation for $L(s, \chi)$ (see [4, §128]) it follows that if $L(s, \chi)$ has a zero $\rho$ with $1/2 < \rho < 1$, then it has a zero $\rho$ with $0 < \rho < 1/2$. As it is known that $L(s, \chi) > 0$ for $1 \leq s$, it suffices to prove $L(s, \chi) > 0$ for $0 \leq s \leq 1/2$. So we take $k = 67$ and $0 \leq s \leq 1/2$. By Theorem 2,

$$L(s, \chi) = \frac{2^{s+1} - 1}{67^s} \left\{ \frac{s^3(s + 1)}{67} \sum_{\chi_3} - \frac{s(s + 1)(s + 2)}{3!(67)^3} \frac{2^{s+3} - 1}{4(2^{s+1} - 1)} \zeta(s + 3) \Sigma_3 + \cdots \right\},$$

where now $\Sigma_M = \sum_{n=1}^{29} \chi(n)(67 - 2n)^M$. For $s > 0$,

$$\zeta(s + 1) - \frac{1}{s} = \sum_{n=1}^{\infty} \frac{1}{n^{s+1}} - \int_{1}^{\infty} \frac{dx}{x^{s+1}} = \sum_{n=1}^{\infty} \left\{ \frac{1}{n^{s+1}} - \int_{n}^{n+1} \frac{dx}{x^{s+1}} \right\} > 0.$$ 

So for $s \geq 0$, $s^3(s + 1) \geq 1$. Also $\Sigma_3 = 67$. So
\[ \frac{s^\xi(s + 1)}{67} \geq 1. \]

For \( 0 \leq s \)
\[ s \frac{2s+2\alpha+1 - 1}{4\alpha(2s+1 - 1)} < \frac{2s}{2 - 2^{-s}} \quad \text{and} \quad \frac{d}{ds} \left( \frac{2s}{2 - 2^{-s}} \right) > 0. \]

So for \( 0 \leq s \leq 1/2 \)
\[ s \frac{2s+2\alpha+1 - 1}{4\alpha(2s+1 - 1)} \leq \frac{2(1/2)}{2 - 2^{-1/2}} < 0.77346. \]

Also
\[ \frac{(s + 1)(s + 2)}{3!} \leq \frac{(3/2) \cdot (5/2)}{3!} = \frac{5}{8}. \]

Since \( \Sigma_3 = -102,845 \), we infer
\[ \frac{s(s + 1)(s + 2)}{3!(67)^s} \frac{2s+3 - 1}{4(2s+1 - 1)} \xi(s + 3) \Sigma_3 \]
\[ \geq - \frac{5}{8} \left( \frac{1}{(67)^s} \right) (0.77346) \xi(3)(102,845) \]
\[ \geq - \frac{5}{8} \left( 0.77346 \right)(1.20206) \frac{102,845}{300,763} > -0.199. \]

Now for \( M \geq 1 \),
\[ \Sigma_M = \left\{ (57 + 8)^M - (57 + 6)^M - (57 + 4)^M + (57 + 2)^M - 57^M \right. \]
\[ + (57 - 2)^M - (57 - 4)^M - (57 - 6)^M + (57 - 8)^M \}
\[ + \left. \{(43 + 4)^M - (43 + 2)^M - 43^M - (43 - 2)^M + (43 - 4)^M \right\} \]
\[ + 37^M + 35^M + \cdots \]
\[ > -57^M + \frac{M(M - 1)}{2!} 57^{M-2} \left\{ 2 \cdot 8^2 - 2 \cdot 6^2 - 2 \cdot 4^2 + 2 \cdot 2^2 \right\} \]
\[ + \frac{M(M - 1)(M - 2)(M - 3)}{4!} 57^{M-4} \left\{ 2 \cdot 8^4 - 2 \cdot 6^4 \right. \]
\[ - 2 \cdot 4^4 + 2 \cdot 2^4 \}
\[ + \cdots \]
\[ > -43^M + \frac{M(M - 1)}{2!} 43^{M-2} \left\{ 2 \cdot 4^4 - 2 \cdot 2^4 \right\} + \cdots \]
\[ \sum_{a=2}^{\infty} \frac{s(s + 1) \cdots (s + 2a)}{(2a + 1)!} \frac{2^{s+2a+1} - 1}{4^a(2^{s+1} - 1)} \xi(s + 2a + 1) \Sigma_{2a+1} \]

\[ \xi(s + 2a + 1) \]

\[ \sum_{a=2}^{\infty} \frac{s(s + 1) \cdots (s + 2a)}{(2a + 1)!} \frac{2^{s+2a+1} - 1}{4^a(2^{s+1} - 1)} \xi(s + 2a + 1) \]

\[ \sum_{a=2}^{\infty} \frac{s(s + 1) \cdots (s + 4)}{5!(67)^{2a+1}} \frac{2^{s+2a+1} - 1}{4^a(2^{s+1} - 1)} \xi(5) \]

\[ \sum_{a=2}^{\infty} \frac{s(s + 1) \cdots (s + 4)}{5!(67)^{2a+1}} \frac{2^{s+2a+1} - 1}{4^a(2^{s+1} - 1)} \xi(5) \]

\[ \sum_{a=2}^{\infty} \frac{s(s + 1) \cdots (s + 4)}{5!(67)^{2a+1}} \frac{2^{s+2a+1} - 1}{4^a(2^{s+1} - 1)} \xi(5) \]

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\[ \sum_{a=2}^{\infty} \frac{s(s + 1) \cdots (s + 4)}{5!(67)^{2a+1}} \frac{2^{s+2a+1} - 1}{4^a(2^{s+1} - 1)} \xi(5) \]
By (1), (2), and (3), for $0 \leq s \leq 1/2$,
\[
L(s, \chi) \geq \frac{2^{s+1} - 1}{6^s} \left\{ 1.000 - 0.199 - 0.638 \right\} \geq \frac{0.163}{(67)^{1/2}} \geq 0.0199.
\]
So $L(s, \chi) > 0$ for $0 \leq s$.

When $\chi(-1) = -1$, Theorem 2 opens up further interesting possibilities. When $s \to 0$, the first term of the series is bounded away from zero, while the remaining terms approach zero. Thus one can always infer $L(s, \chi) > 0$ for $0 \leq s \leq \epsilon$, where $\epsilon$ depends on $k$. Even for $\epsilon$ as small as $A/\log k$, this would be a very worthwhile result, as remarked at the beginning of the paper.

For another possibility, let $s = 0$ and $-2$ in Theorem 2, and evaluate $L(0, \chi)$ and $L(-2, \chi)$ by the functional equation. We infer the known result

(4) \[ L(1, \chi) = \frac{\pi}{k^{3/2}} \Sigma_1 \]

and the result

(5) \[ L(3, \chi) = \frac{\pi^3}{6k^{7/2}} \left\{ k^2 \Sigma_1 - \Sigma_2 \right\}. \]

From these follow

(6) \[ \Sigma_3 = k^{7/2} \left\{ \frac{L(1, \chi)}{\pi} - \frac{6L(3, \chi)}{\pi^3} \right\}. \]

This gives

\[ \Sigma_3 \geq -k^{7/2} \frac{6L(3, \chi)}{\pi^3}. \]

If one could prove independently any appreciably better result, one could derive a sensational inequality for $L(1, \chi)$. For instance, if one could prove

\[ \Sigma_3 \geq -k^{7/2} \frac{4}{\pi^3} \geq -k^{7/2} \frac{5L(3, \chi)}{\pi^3}, \]

one could get by (6)

\[ L(1, \chi) > \frac{L(3, \chi)}{\pi^2}. \]

Another possibility is that one can perhaps derive some connec-
tion between $\Sigma_1$ and $\Sigma_3$. For instance, if one could prove

$$\Sigma_3 \geq -k^2 \log k \Sigma_1,$$

then by (4) and (6), we could infer

$$L(1, \chi) > \frac{6L(3, \chi)}{\pi^2(1 + \log k)}.$$ 

Even this would be a very worthwhile result, since the best known at present is, by Siegel’s Theorem,

$$L(1, \chi) > \frac{L(3, \chi)}{k^\epsilon}$$

for $\epsilon > 0$ and large $k$.

**Bibliography**


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