

# CLASSIFICATION OF 2-MANIFOLDS WITH SINGULAR POINTS

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**1. Introduction.** By a *closed 2-manifold*, or simply a 2-manifold, we mean here a two-dimensional connected finite simplicial complex every point of which has a neighborhood homeomorphic to a circular disk, that is, the interior of a circle. If there is a point not having the latter property, we call it a *singular point* of it.

In this paper, we shall give a complete classification (§2) and some properties (§3) of 2-manifolds with a single singular point. Obviously one may get one such geometrical figure by identifying certain points of a 2-manifold or several ones together. Conversely, we shall show that every such figure may be obtained in such a manner (see (2.3)).

In §4, we generalize these results to 2-manifolds with any number of singularities.

**2. The classification.** The classification lies in the investigation of the nature of the neighborhoods of the singular point. Let  $\mathfrak{M}^2$  have its singular point at 0. We first establish the following lemma.

**LEMMA (2.1).** *Any neighborhood of 0 is homeomorphic to a finite number, say  $p$ , of circular disks with all their centers identified. We call it a  $p$ -bundle and call 0 its center; and the boundaries of these  $p$  disks are simply said to be the boundary of the  $p$ -bundle.*

**PROOF.** Consider a simplicial subdivision  $\mathfrak{R}^2$  of  $\mathfrak{M}^2$ . We first note that 0 must be a vertex of  $\mathfrak{R}^2$ . For, if 0 were an inner point of a 2-simplex, then 0 could not belong to any other simplex and hence would be an ordinary point; and if it were an inner point of a 1-simplex, then all the points of this 1-simplex would be singular points for the same reason. It is also evident that 0 cannot be a vertex of a 1-simplex unless it is a vertex of a 2-simplex.

Let 0 be a vertex of a 2-simplex  $\mathfrak{R}^2$ . Then there must be many 2-simplexes including  $\mathfrak{R}^2$  forming a circular disk surrounding 0, as otherwise there would be two edges of singularities. Besides, 0 must be a vertex of another 2-simplex, say  $\mathfrak{R}'^2$ , by noting that 0 is a singular point. Hence we get another circular disk consisting of

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<sup>1</sup> We use the notation  $\mathfrak{M}$  instead of a dot directly over  $\mathfrak{M}$  for typographical convenience.

2-simplexes surrounding 0. In such a way, finally, we obtain  $p$  such circular disks (since  $\cdot\mathcal{M}^2$  is finite), and the lemma is proved.

We say  $\cdot\mathcal{M}^2$  has a singularity of order  $p$  at 0.  $\cdot\mathcal{M}^2 - (0)$  in general is not connected and consists of  $n$  components  $\mathcal{R}_i^2$  ( $i=1, \dots, n$ ). We name  $\mathcal{R}_i^2 + (0)$  a sheet of  $\cdot\mathcal{M}^2$ . Then  $\cdot\mathcal{M}^2$  is the sum of these sheets with the identification of 0. Moreover, each circular disk of 0 belongs wholly to one and only one sheet. Let us denote these sheets by  $\cdot\mathcal{M}_i^2 = \mathcal{R}_i^2 + (0)$  ( $i=1, \dots, n$ ), then

$$(1) \quad \cdot\mathcal{M}^2 = \sum_{i=1}^n \cdot\mathcal{M}_i^2,$$

$$(2) \quad p = \sum_{i=1}^n p_i,$$

where  $p_i$  is the order of 0 in  $\cdot\mathcal{M}_i^2$  ( $p_i=1$  in case  $\cdot\mathcal{M}_i^2$  itself is a 2-manifold).

Hence it is sufficient for us to consider  $\cdot\mathcal{M}_i^2$  separately. But the structure of  $\cdot\mathcal{M}_i^2$  is quite clear; for, if we take away the  $p_i$  circular disks surrounding 0, the rest is a bounded 2-manifold with  $p_i$  holes, the classification of which is already well known.<sup>2</sup> Therefore we get:

**THEOREM (2.2).** *Any 2-manifold with one singularity may be decomposed into the form (1), where  $\cdot\mathcal{M}_i^2$  are sheets, that is, bounded 2-manifolds with  $p_i$  holes adjoined with a  $p_i$ -bundle having its boundary identified with the boundaries of these holes; and all the centers of these bundles are to be identified.*

We may, however, consider the  $p$ -bundle separately as  $p$  circular disks and identify each of their circumferences with each of the boundaries of the holes. Thus we get 2-manifolds, and then identify the  $p$  centers. Hence we obtain:

**THEOREM (2.3).** *Every 2-manifold with one singular point is the sum of a finite number of 2-manifolds each with some points identified all together.*

The preceding two theorems lead us to obtain a 2-manifold with a single singular point from bounded and closed 2-manifolds respectively. In practice the latter is much more useful than the former.

**3. Simple properties.** The most evident property of 2-manifolds with one singularity is contained in the following theorem.

**THEOREM (3.1).**  *$\cdot\mathcal{M}^2$  and  $\cdot\mathcal{M}^{*2}$  are homeomorphic if and only if they*

<sup>2</sup> Cf., for example, Seifert-Threlfall, *Lehrbuch der Topologie*, 1934, §40.

have the same structures, hence necessarily, after suitably arranging their sheets,

$$(3) \quad (p_i) = (p_i^*)$$

(consequently  $n = n^*$ ,  $p = p^*$ ).

But we should notice that (3) is not a sufficient condition for  $\cdot\mathcal{M}^2$  and  $\cdot\mathcal{M}^{*2}$  to be homeomorphic, since the corresponding 2-manifolds constituting them may not be homeomorphic. We also note that the property of orientability is preserved:

**THEOREM (3.2).** *A 2-manifold with one singularity is orientable if and only if all the 2-manifolds constituting it are orientable.*

Now we come to prove the important theorem:

**THEOREM (3.3).** *The (integral) homology group  $\mathfrak{h}$  of any dimension of a 2-manifold with one singularity is the direct sum of those of its sheets. The homology group of a sheet  $\cdot\mathcal{M}_i^2$  is the same as that of the 2-manifold  $\mathcal{M}_i^2$  except for dimension 1, where  $\mathcal{M}_i^2$  is the 2-manifold which is the same as  $\cdot\mathcal{M}_i^2$  but without those  $p_i$  points to be identified in  $\cdot\mathcal{M}_i^2$  being identified in  $\mathcal{M}_i^2$ . For dimension 1, if  $\cdot\mathcal{M}_i^2$  and  $\mathcal{M}_i^2$  have the homology groups  $\cdot\mathfrak{h}_i^1$  and  $\mathfrak{h}_i^1$  respectively, then*

$$(4) \quad \cdot\mathfrak{h}_i^1 = \mathfrak{h}_i^1 + (p_i - 1)g$$

where  $kg$  represents the direct sum of  $k$  free cyclic groups  $g$ .

**PROOF.** Let us consider the last statement only as the others are obvious. In case  $p_i = 1$ , (4) is trivial.

Suppose  $0_1, \dots, 0_{p_i}$  are the points of  $\mathcal{M}_i^2$  which are to be identified in  $\cdot\mathcal{M}_i^2$ . Take a sufficiently small simplicial subdivision of  $\mathcal{M}_i^2$  such that all these points are vertices, and it induces a simplicial subdivision on  $\cdot\mathcal{M}_i^2$ . Then any 1-cycle  $Z^1$  on  $\cdot\mathcal{M}_i^2$  is either a 1-cycle on  $\mathcal{M}_i^2$  or a broken line joining two 0's, say  $0_j$  and  $0_k$ . In the latter case  $Z^1$  is neither homologous nor division homologous to zero. For if  $mZ^1 \sim 0$  ( $m \neq 0$ ), then there would exist a 2-chain  $C^2$  whose boundary  $\partial C^2 = mZ^1$  and thence

$$\partial\partial C^2 = m\partial Z^1 = m(\pm 0_j \pm 0_k) \neq 0,$$

which is a contradiction. Hence  $Z^1$  as an element of  $\cdot\mathfrak{h}_i^1$  generates a free cyclic group.

We then join  $0_1$  to the other 0's and get  $p_i - 1$  broken lines, each of which generates a free cyclic group since no two of them are homologous or division homologous to zero by the same reason.

Any broken line between  $0_j$  and  $0_k$  may be replaced by an algebraic sum of two broken lines starting from  $0_1$  and ending in  $0_j$ ,  $0_k$  respectively, and a 1-cycle through these three points on  $\mathfrak{M}_1^2$  will be discussed below.

A 1-cycle on  $\mathfrak{M}_1^2$  not passing through any 0 is not influenced in constructing  $\cdot\mathfrak{M}_1^2$ , while one passing any 0, say  $0_k$ , may be modified by omitting the two edges through it and adding the third edge of the 2-simplex that is incident with  $0_k$  as well as the two edges through  $0_k$  (see (2.1)). Hence the homology classes made by the 1-cycles on  $\mathfrak{M}_1^2$  are unchanged on  $\cdot\mathfrak{M}_1^2$ . Thus (3.3) is established.

COROLLARY (3.4).

$$\cdot\mathfrak{h}^1 = \sum_{i=1}^n \cdot\mathfrak{h}_i^1 + (p - n)g.$$

This shows us a method for constructing a 2-complex with any preassigned Betti number whatever.

Analogously, we have the following theorem.

THEOREM (3.5). *The fundamental group of a 2-manifold with one singularity is the free product of those of its sheets, and the fundamental group  $\mathfrak{f}_i$  of a sheet  $\cdot\mathfrak{M}_i^2$  is the free product of the fundamental group  $\mathfrak{f}_i$  of  $\mathfrak{M}_i^2$  and  $p_i - 1$  free cyclic groups.<sup>3</sup>*

4. **Generalizations.** By the finiteness of a 2-manifold it is evident that the number of singular points on it, if any, is finite. Let  $'0, ''0, \dots, {}^{(m)}0$  be the only singular points on a 2-manifold  $\cdot\mathfrak{M}^2$ .

Lemma (2.1) is valid for each  ${}^{(j)}0$  ( $j=1, \dots, m$ ) and we may speak of the order at  ${}^{(j)}0$ , say  ${}^{(j)}p$ . The generalized Theorems (2.2) and (2.3) have their natural forms, the latter of which we state as follows.

THEOREM (4.1).  *$\cdot\mathfrak{M}^2$  is the sum of a finite number of 2-manifolds  $\cdot\mathfrak{M}_i^2$  ( $i=1, \dots, n$ ) on which  ${}^{(j)}p_i$  points are identified to the point  ${}^{(j)}0$  ( $j=1, \dots, m$ ).*

Moreover,

$$(5) \quad p_i = \sum_{j=1}^m {}^{(j)}p_i, \quad {}^{(i)}p = \sum_{i=1}^n {}^{(i)}p_i, \quad p = \sum_{i=1}^n p_i = \sum_{j=1}^m {}^{(j)}p,$$

where  $p_i$  and  $p$  are defined as the orders of  $\cdot\mathfrak{M}_i^2$  and  $\cdot\mathfrak{M}^2$  respectively.

<sup>3</sup> We may first prove (3.5) and so (3.3) follows immediately by a relation between the homology group and the fundamental group, cf. *ibid.* p. 173.

Theorem (3.1) now takes the form:

THEOREM (4.2).  $\mathcal{M}^2$  and  $\mathcal{M}^{*2}$  are homeomorphic if and only if they have the same structures, hence necessarily, after suitably arranging their sheets and the order of their singular points,

$$(6) \quad ({}^i p_i) = ({}^i p_i^*)$$

(consequently  $m = m^*$ ,  $n = n^*$ ,  $p = p^*$ ).

Theorem (3.2) is true in its original form. We establish now the following theorem.

THEOREM (4.3). The homology group  $\cdot h_i^1$  of  $\cdot \mathcal{M}_i^2$  is given by

$$(7) \quad \cdot h_i^1 = h_i^1 + (p_i - m)g,$$

where  $h_i^1$  is the homology group of  $\mathcal{M}_i^2$ , the 2-manifold which is the same as  $\cdot \mathcal{M}_i^2$  but without any points being identified.

PROOF. For each  $j$  we consider the  $({}^j p_i)$  points to be identified to  $({}^j)0$  as in the proof of (3.3), that is, join broken lines from one of them to all the others. They are 1-cycles on  $\cdot \mathcal{M}_i^2$ , each of which generates a free cyclic group in  $\cdot h_i^1$  and any two of which are neither homologous nor division homologous to zero on  $\cdot \mathcal{M}_i^2$ . Any 1-cycle on  $\cdot \mathcal{M}_i^2$  may be replaced by an algebraic sum of these broken lines and a 1-cycle on  $\mathcal{M}_i^2$ . Hence by the same reason as in (3.3), from the first equation of (5) we have (7), and thus the theorem is proved.

In order to get the homology group  $\cdot h^1$  of  $\cdot \mathcal{M}^2$ , we again introduce a lemma which may be readily proved.

LEMMA (4.4). If  $\mathcal{R}$  is a connected simplicial complex of any dimension and  $P_1, \dots, P_k$  are  $k$  arbitrary distinct points on it, and  $\mathcal{R}^*$  is the complex made by  $\mathcal{R}$  in addition with the 1-simplexes  $(P_1 P_2), (P_1 P_3), \dots, (P_1 P_k)$  (not in  $\mathcal{R}$ ); then

$$(8) \quad h^{*1} = h^1 + (k - 1)g,$$

or

$$(8)' \quad h^1 = h^{*1} - (k - 1)g,$$

where  $h^1$  and  $h^{*1}$  are homology groups of  $\mathcal{R}$  and  $\mathcal{R}^*$  respectively and the minus sign indicates a difference group.

Eventually, we have the following theorem.

THEOREM (4.5). The homology group  $\cdot h^1$  of  $\cdot \mathcal{M}^2$  may be written as

$$(9) \quad \cdot h^1 = \sum_{i=1}^n h_i^1 + (p - n - m + 1)g.$$

PROOF. In constructing 1-simplexes  $(\cdot 0'0)$ ,  $\dots$ ,  $(\cdot 0^{(m)}0)$  (not belonging to  $\cdot \mathcal{M}^2$ ), we get  $\cdot \mathcal{M}^{*2}$ ,  $\cdot \mathcal{M}_1^{*2}$ ,  $\dots$  as in the lemma. By (7) and (8), we have

$$(10) \quad \cdot h_i^{*1} = h_i^1 + (p_i - 1)g,$$

where  $\cdot h_i^{*1}$  is the homology group of  $\cdot \mathcal{M}_i^{*2}$ . The newly constructed simplexes form a connected 1-complex whose 1-dimensional homology group contains the identity only. Therefore from a famous theorem (cf. Seifert-Threlfall, p. 179), by (5) we get

$$(11) \quad \cdot h^{*1} = \sum_{i=1}^n h_i^1 + (p - n)g,$$

where  $\cdot h^{*1}$  is the homology group of  $\cdot \mathcal{M}^{*2}$ . Therefore (9) is finally established in virtue of (11) and (8)'.

Theorem (3.5) may be extended analogously.

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## A NOTE ON EQUICONTINUITY

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During a recent seminar discussion of his paper *Transitivity and equicontinuity* [1],<sup>1</sup> W. H. Gottschalk proposed the following question:

“Is the center of every algebraically transitive group of homeomorphisms on a compact metric space equicontinuous?”

An affirmative answer to the above question is given in this note.

**1. Definitions.** We let  $X$  and  $Y$  be compact metric spaces and let  $d$  be the metric for  $Y$ .

A set  $F$  of functions on  $X$  into  $X$  is *algebraically transitive* if corresponding to each pair  $p$  and  $q$  of points in  $X$  there exists  $f \in F$  such that  $f(p) = q$ .

A sequence  $[g_n]$  of functions on  $X$  into  $Y$  converges to a function

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<sup>1</sup> Numbers in brackets refer to the bibliography at the end of the paper.