

**ON THE NUMBER OF POSITIVE INTEGERS LESS THAN x
AND FREE OF PRIME DIVISORS GREATER THAN x^c**

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Dr. Chowla recently raised the following question regarding the number of positive integers less than x and free of prime divisors greater than x^c , which number is here denoted by $f(x, c)$: For every fixed positive c , is $\liminf_{x \rightarrow \infty} f(x, c)/x > 0$?

This paper, while incidentally answering this question in the affirmative, proves more, in fact the best¹ possible result in this direction, namely:

THEOREM A. *A function $\phi(c)$ defined for all $c > 0$ exists such that*

- (1) $\phi(c) > 0$ and is continuous for $c > 0$;
- (2) for any fixed c

$$f(x, c) = x\phi(c) + O(x/\log x)$$

where the "O" is uniform for c greater than or equal to any given positive number.

Notation. The following symbols are used for the entities mentioned against them:

p, p_r : any prime.

$S(x, p)$: the set of integers less than x each divisible by p and free of prime divisors greater than p .

$T(x, p)$: the set of integers less than x each free of prime divisors greater than p .

$N[K]$: the number of members of K , where K denotes any finite set of integers.

$F(t)$: $\sum_{p \leq t} (1/p)$ where p runs through primes.

Preliminary lemmas.

LEMMA I. *For $c \geq 1$, the theorem is true, and*

$$f(x, c) = x\phi(c) + O(1).$$

PROOF. This is obvious. In fact, for these values of c , $\phi(c) = 1$, and

$$|f(x, c) - x\phi(c)| \leq 1.$$

LEMMA II. *If $p_1 \neq p_2$, the sets $S(x, p_1)$ and $S(x, p_2)$ are distinct, and*

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¹ The "O" of the theorem cannot be improved upon, as will be seen in the sequel.

$$N[S(x, p)] = N[T(x/p, p)].$$

PROOF. This is obvious, from the unique factorization theorem.

LEMMA III. *The number of primes less than or equal to x is $O(x/\log x)$ and $F(x) = \log \log x + b + O(1/\log x)$ where b is constant.*

This is well known and is an “elementary theorem” in the theory of primes.

LEMMA IV. *If $0 < c_1 \leq 1$, and Theorem A is true for $c \geq c_1$, then it is true for $c \geq c_1/(1+c_1)$.*

PROOF. By hypothesis $\phi(c)$ is defined for $c \geq c_1$, and

$$\phi(c) > 0 \text{ and is continuous for } c \geq c_1,$$

$$(3) \quad \begin{aligned} f(x, c) &= x\phi(c) + O(x/\log x) \text{ uniformly for } c_1 \leq c \leq 1, \\ f(x, c) &= x\phi(c) + O(1) \text{ for } c \geq 1, \end{aligned} \quad \text{by Lemma 1.}$$

Also, obviously,

(4) $\phi(c)$ is bounded and monotonic increasing, though possibly not strictly so, for $c \geq c_1$.

Let now

$$(5) \quad c_2 = c/(1+c_1) \text{ and } c_2 \leq d \leq c_1 \text{ (obviously } c_2 < c_1).$$

Now

$$\begin{aligned} f(x, c_1) - f(x, d) &= \sum_{x^d < p \leq x^{c_1}} N[S(x, p)] && \text{(by Lemma II)} \\ &= \sum_{x^d < p \leq x^{c_1}} N\left[T\left(\frac{x}{p}, p\right)\right] && \text{(by Lemma II)} \\ &= \sum_{x^d < p \leq x^{c_1}} f\left(\frac{x}{p}, \frac{\log p}{\log x - \log p}\right) \\ &= \sum_{x^d < p \leq x^{c_1}} \left\{ \frac{x}{p} \phi\left(\frac{\log p}{\log x - \log p}\right) \right\} \\ &\quad + \sum_{p \geq x^{1/2}} O(1) + \sum_{p < x^{1/2}} O\left(\frac{x}{p \log(x/p)}\right) \end{aligned}$$

by (3) and (5), since in this range

$$(6) \quad \frac{\log p}{\log x - \log p} > \frac{d}{1-d} \geq \frac{c_2}{1-c_2} = c_1.$$

Hence

$$(6a) \quad f(x, c_1) - f(x, d) = x \int_{x^d}^{x^{c_1}} \phi\left(\frac{\log t}{\log x - \log t}\right) dF(t) + O\left(\frac{x}{\log x}\right)$$

by Lemma III, where the “ O ” is uniform with respect to d in virtue of (3) and (6), and the Riemann-Stieltjes integral on the right exists in virtue of (4) and the continuity of $\phi(c)$ for $c \geq c_1$. Using integration by parts for the integral and using (3) and (4) and Lemma III, we obtain (see Note 1a)

$$(7) \quad \begin{aligned} f(x, c_1) - f(x, d) &= x \int_{x^d}^{x^{c_1}} \phi\left(\frac{\log t}{\log x - \log t}\right) \frac{dt}{t \log t} + O\left(\frac{x}{\log x}\right) \\ &\quad \text{(uniformly for } c_2 \leq d \leq c_1) \\ &= x \int_d^{c_1} \phi\left(\frac{u}{1-u}\right) \frac{du}{u} + O\left(\frac{x}{\log x}\right) \\ &\quad \text{(uniformly for } c_2 \leq d \leq c_1). \end{aligned}$$

This, together with (3), proves the existence of $\phi(c)$ for $c \geq c_1/(1+c_1)$, though it is not yet clear whether $\phi(c) > 0$ in $c_1/(1+c_1) \leq c \leq c_1$, and is continuous therein.

Now, let

$$(8) \quad \frac{c_1}{1+c_1} \leq d_1 < d_2 \leq c_1.$$

Then (7) gives

$$\begin{aligned} \frac{f(x, d_2) - f(x, d_1)}{x} &= \int_{d_1}^{d_2} \phi\left(\frac{u}{1-u}\right) \frac{du}{u} + O\left(\frac{1}{\log x}\right) \\ &\rightarrow \int_{d_1}^{d_2} \phi\left(\frac{u}{1-u}\right) du \quad \text{as } x \rightarrow \infty. \end{aligned}$$

Hence

$$(9) \quad \phi(d_2) - \phi(d_1) = \int_{d_1}^{d_2} \phi\left(\frac{u}{1-u}\right) \frac{du}{u} > 0,$$

by (3), since in the range $d_1 < u < d_2$, we have $u/(1-u) > c_1$.

This shows that $\phi(d_2) \neq 0$ for any d_2 of the kind specified in (8); for, if $\phi(d_2)$ were zero, then obviously $\phi(d_1)$ would be zero and their difference also would be so, contrary to (9). Also, obviously $\phi(d_2) \geq 0$. Hence

$$(10) \quad \phi(c) > 0 \quad \text{for } c > c_1/(1+c_1).$$

Also by (9) and the hypothesis

$$(11) \quad \phi(c) \text{ is continuous for } c \geq c_1/(1 + c_1).$$

Using now the results (7), (10), and (11) and repeating the above argument with $c_1/(1+c_1)$ in place of c_1 , and noting that the positive-ness of $\phi(c_1)$ is not needed in the above argument, it follows that $\phi(c_1/(1+c_1)) > 0$.

This completes the proof of the lemma.

PROOF OF THE THEOREM. Lemma I holds for $c \geq 1$ and Lemma IV for $0 < c < 1$, if we note that the hypotheses of the lemma are satisfied for $c_1=1$, and hence by the result of the lemma for $c_1=1/2$ and hence by induction for $c_1=1/n$ (n positive integral), and that $1/n \rightarrow 0$ as $n \rightarrow \infty$.

COROLLARY. For $0 < c_1 < c_2 \leq 1$,

$$\phi(c_2) - \phi(c_1) = \int_{c_1}^{c_2} \phi\left(\frac{u}{1-u}\right) \frac{du}{u}$$

(see also Note 1b).

PROOF. A finite set of numbers, $d_0, d_1, d_2, \dots, d_n$, obviously exists such that

$$c_1 = d_0 < d_1 < d_2 < \dots < d_n = c_2;$$

and

$$d_r \geq \frac{d_{(r+1)}}{1 + d_{(r+1)}} \quad (r = 0, 1, 2, \dots, n - 1).$$

To each of these intervals (d_r, d_{r+1}) apply the result (9) and add; the corollary follows at once.

REMARKS. The "O" of the theorem cannot be improved upon. This can be seen as follows:

Obviously $f(x, 1) = x + O(1)$ and

$$f(x, 1) - f\left(x, \frac{1}{2}\right) = \sum_{x^{1/2} < p \leq x} \left[\frac{x}{p} \right] = x \sum_{x^{1/2} < p \leq x} \left(\frac{1}{p} \right) - G(x)$$

where $G(x) = \sum_{x^{1/2} < p \leq x} \{x/p\}$ and $\{y\}$ denotes fractional part of y .

Hence $f(x, 1/2) = x(1 - \log 2) + o(x/\log x) + G(x)$ by the prime number theory (see Note 1c). But $G(x) > kx/\log x$ where k is a fixed positive number, as can be easily seen from the prime number theory.

NOTE 1. (a) The deduction of (7) from (6a) is based on the following:

We observe that the Riemann-Stieltjes integral

$$\begin{aligned} \int_{x^d}^{x^{d_1}} \phi\left(\frac{\log t}{\log x - \log t}\right) dF(t) &= \left[F(t)\phi\left(\frac{\log t}{\log x - \log t}\right) \right]_{x^d}^{x^{d_1}} \\ &\quad - \int_{x^d}^{x^{d_1}} F(t) d\phi \\ &= \left[(\log \log t + b)\phi\left(\frac{\log t}{\log x - \log t}\right) \right]_{x^d}^{x^{d_1}} \\ &\quad - \int_{x^d}^{x^{d_1}} (\log \log t + b) d\phi + R \end{aligned}$$

which, by the substitution $t = x^u$ performed after integration by parts of the integral on the right, equals

$$\int_d^{d_1} \phi\left(\frac{u}{1-u}\right) \frac{du}{u} + R$$

where, by Lemma III,

$$R = O\left(\frac{1}{d \log x}\right) - O\left(\frac{1}{d \log x}\right) - \int_{x^d}^{x^{d_1}} O\left(\frac{1}{\log t}\right) d\phi$$

where, on account of (4), the integral also is $O(d^{-1} \log^{-1} x)$.

(b) The integral equation for $\phi(c)$ may be used for successive computation of the function. Starting with $\phi(c) = 1$ for $c \geq 1$, one observes that $c/(1-c) \geq 1$ for $1/2 \leq c < 1$ so that

$$\phi(c) = 1 - \int_c^1 \frac{du}{u} = 1 + \log c \quad \text{for } 1/2 \leq c \leq 1.$$

This result can be used to compute $\phi(c)$ on the interval $(1/3, 1/2)$ and so on; and an easy induction shows that

$$\phi(c) = 1 + \sum_{r=1}^{\infty} (-1)^r \psi_r(c), \quad c > 0,$$

where $\psi_1(c) = \int_c^1 du/u$ for $c \leq 1$, and $\psi_1(c) = 0$ for $c > 1$, and

$$\psi_r(c) = \int_c^1 \psi_{r-1}\left(\frac{u}{1-u}\right) \frac{du}{u}, \quad \text{for } c > 0.$$

One notes that $\psi_r(c) = 0$ for $c \geq 1/r$ (so that the infinite series is actually a finite sum) and also that the functions $\psi_r(c)$, $r \geq 2$, are not elementary functions.

(c) From the known result $\pi(x) = x/\log x + x/\log^2 x + o(x/\log^2 x)$ follows

$$\begin{aligned} \sum_{x^{1/2} < p \leq x} \frac{1}{p} &= \int_{x^{1/2}}^x \frac{1}{t} d\pi(t) = \frac{\pi(x)}{x} - \frac{\pi(x^{1/2})}{x^{1/2}} + \int_{x^{1/2}}^x \frac{\pi(t)}{t^2} dt \\ &= -\frac{1}{\log x} + o\left(\frac{1}{\log x}\right) \\ &\quad + \int_{x^{1/2}}^x \left[\frac{t}{\log t} + \frac{t}{\log^2 t} + o\left(\frac{t}{\log^2 t}\right) \right] \frac{dt}{t^2} \\ &= \log 2 + o\left(\frac{1}{\log x}\right). \end{aligned}$$

NOTE 2. From the corollary to the theorem follows, for $0 < c_1 < c_2 \leq c_1/(1-c_1)$ and $c_2 \leq 1$,

$$\phi(c_2) - \phi(c_1) \geq \phi(c_2) \log c_2/c_1 > \phi(c_2)(1 - c_1/c_2)$$

from which follows $\phi(c_2)/c_2 > \phi(c_1)/c_1$, whence, arguing as in the proof of the corollary, one sees that $\phi(c)/c$ is a strictly monotonic increasing function in $0 < c \leq 1$.

$\phi(c)$ has other interesting properties, which will be published shortly. One such is that for every fixed n , $\phi(c)/c^n \rightarrow 0$ as $c \rightarrow +0$. This result, together with the now obvious result

$$\lim_{x \rightarrow \infty} \frac{f(x, c)}{x} = \phi(c) > 0, \quad c \text{ fixed and positive,}$$

was communicated to Dr. Chowla in August of 1947. I understand from his reply that Vijayaraghavan already was in possession of a proof of his (Chowla's) conjecture that

$$\liminf_{x \rightarrow \infty} \frac{f(x, c)}{x} > 0, \quad \text{for } c \text{ positive and fixed.}$$

In conclusion, I wish to thank the referees for their suggestions which have led to the clarification and additions of content contained in Note 1.

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