

TOPOLOGICAL METHODS IN THE THEORY OF LEBESGUE AREA

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1. **Introduction.** In conformity with the title of this address, the principal objective is the consideration of topological techniques which have been employed in the theory of surface area initiated by Lebesgue [15]¹ in his thesis.

It may appear strange that topology is required at all in a discipline classified in the field of analysis—certainly Lebesgue in 1902 gave little hint of what was to come. On the other hand, it cannot be claimed that this sort of thing is an isolated phenomenon, and perhaps one may be permitted to argue that one of the beauties of mathematics is the presence of such mixtures between fields.

Historically speaking, it was de Geöcze who called attention to certain phenomena—in a series of highly significant papers written during the years 1908 to 1914—and, in attempting to fit these phenomena into the framework of the theory, first noticed the need for topology. (For a partial list of these papers see de Geöcze [12]. A more detailed bibliography at this point, and elsewhere throughout the paper, is found in Radó [26].)

In discussing this matter let it be said, even at the risk of considerable over-simplification, that *the significant problems in Lebesgue area have their source in the attempt to find 2-dimensional analogues for four classical results in the theory of length*. In any event, with an eye on cohesion, it is here planned to consider only those topological techniques which have bearing on what might be called the *analogy problem*.

In consequence of this general plan of attack it is not only appropriate to sketch these classical results, but fortunately in so doing one automatically gains a measure of intuition by analogy and adequate motivation is provided for the 2-dimensional theory.

Before plunging into the discussion, however, a remark should be made as to the manner of presentation in contradistinction to the plan of attack. Matters of analogy leading to intuitive understanding are considered important enough to develop so as to be practically self-contained. (For example, during the first part of this paper every

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¹ Numbers in brackets refer to the bibliography at the end of the paper.

effort will be made to define the principal concepts involved.) Consequently, it is to be hoped that a substantial portion of the subject matter will be followed readily by all. This plan must, of course, be abandoned in those sections dealing with specific topological techniques—this is certainly not the place to consider such matters in any detail.

2. Coordinate functions. For the purpose of considering the previously mentioned four classical results for curves, suppose f is a mapping, that is, a continuous transformation, from the unit interval X^1 into Euclidean 3-space E^3 . It is assumed that *coordinate axes* Y^1 , Y^2 , and Y^3 have been selected in E^3 . If $p^i: E^3 \rightarrow Y^i$ is the notation for the *perpendicular projection* of E^3 onto Y^i , while f^i is the composition $p^i f$, then $f^i: X^1 \rightarrow Y^i$, $i = 1, 2, 3$, and in analysis the mapping f is usually given by the so-called *coordinate functions*:

$$y^1 = f^1(x), \quad y^2 = f^2(x), \quad y^3 = f^3(x), \quad x \in X.$$

3. Classical formula. The classical integral formula $\int_{X^1} [(J^1)^2 + (J^2)^2 + (J^3)^2]^{1/2} dx$, where $J^i = df^i/dx$, $i = 1, 2, 3$, is to be designated by $I^1(f)$. This formula will *not* generally provide the length of the “*curve given by f .*” Indeed, even if Lebesgue integration is used—and this will invariably be the case—the formula will generally be a meaningless collection of marks, since there is no guarantee that the integrand exists almost everywhere and is summable. In this connection it is to be understood, however, that *if the symbol $I^1(f)$ appears in a statement then it has meaning*; in other words, the integrand in question does exist almost everywhere and, indeed, is summable.

4. Representation. Before going any farther, it is advisable to make some comment on the phrase “*curve given by f* ” used in §3.

The mapping f is simply a *representation* for a curve and the concept of changing the representation is common, even in elementary mathematics, though it is subject to many interpretations. For a number of reasons the most convenient interpretation for present purposes is due to Fréchet [11]. Accordingly $f: X^1 \rightarrow E^3$ and $g: X^1 \rightarrow E^3$ are said to *represent the same curve*, or, to use the modern terminology, are said to be *Fréchet equivalent* (notation: $f \sim g$), if and only if for every $\epsilon > 0$ there is a homeomorphism $h_\epsilon: X^1 \approx X^1$ such that $d\{f(x), gh_\epsilon(x)\} < \epsilon$, $x \in X^1$. A formal definition for a curve can now be offered in terms of this equivalence relation. (See Youngs [31] where the matter is considered in detail for both curves and surfaces.)

It is well known that the length of a curve is initially defined in terms of a *given representation* as the limit of the lengths of any

sequence of inscribed polygonal mappings which converge uniformly to the given representation. It is easy to see, however, that the length of a curve is—as it should be—*independent* of the representation. (For greater detail see Radó [26].) The notation $L^1(f)$ is here employed for the length of the curve given by the mapping $f: X^1 \rightarrow E^3$.

5. Classical theorems for curves and future criteria for surfaces.

It is now possible to state three of the promised classical results:

$C_1: L^1(f) < \infty \leftrightarrow f^i$ is BV (= of bounded variation), $i = 1, 2, 3$.

$C_2: L^1(f) < \infty \rightarrow L^1(f) \geq I^1(f)$, and $L^1(f) = I^1(f) \leftrightarrow f^i$ is AC (= absolutely continuous), $i = 1, 2, 3$. (In conformity with the convention in §3 the first statement means: $L^1(f) < \infty \rightarrow I^1(f)$ exists and $L^1(f) \geq I^1(f)$.)

$C_3: L^1(f) < \infty \rightarrow$ there is a representation $g \sim f$ such that $I^1(g) = L^1(g) = L^1(f)$.

6. Interval function. It would be presumptuous to define the concepts BV and AC except for the fact that a record of these definitions affords both an insight into the 2-dimensional case and a considerable economy in the sequel.

For convenience suppose $\phi: X^1 \rightarrow E^1$ is a mapping from the unit interval into a line. (The mapping ϕ is of the same type, therefore, as the coordinate functions f^1, f^2 , and f^3 of §2.)

The key to the definitions of BV and AC is a certain *interval function* now to be considered. Suppose that r is a closed interval $a \leq x \leq b$ in X^1 , while $\dot{r} = a \cup b$. Then $v_\phi(r)$, the interval function in question, is simply the length of the interval from $\phi(a)$ to $\phi(b)$ in E^1 .

7. An alternative interpretation. For the sake of insight into the 2-dimensional case another interpretation of $v_\phi(r)$ is important.

Consider the components (= maximal connected subsets) of $E^1 - \phi(\dot{r})$, and to avoid trivialities assume there is a bounded component K . Then

$$v_\phi(r) = |K|$$

where $|K|$ is the 1-dimensional Lebesgue measure of K ; in this case, simply the length of the open interval K .

Let Γ be the family of those mappings $\gamma: r \rightarrow E^1$ which agree with ϕ on \dot{r} . Then

$$\gamma(r) \supset K \cup \phi(\dot{r}), \quad \gamma \in \Gamma,$$

but there is a mapping $\gamma_0 \in \Gamma$ (for example the unique linear mapping in Γ) such that

$$\gamma_0(r) = K \cup \phi(\dot{r}).$$

Hence

$$\bigcap_{\Gamma} \gamma(r) = \gamma_0(r).$$

Therefore, $[E^1 - \phi(t)] \cap \bigcap_{\Gamma} \gamma(r) = [E^1 - \phi(t)] \cap \gamma_0(r) = \gamma_0(r) - \phi(t) = K$, and one has the interpretation

$$v_{\phi}(r) = | [E^1 - \phi(t)] \cap \bigcap_{\Gamma} \gamma(r) |.$$

8. Bounded variation and absolute continuity. If R is a closed interval in X , let $\sigma(R)$ be the notation for a subdivision of R ; that is, a *finite* collection of r 's, whose union is R and whose interiors are disjoint in pairs. Define

$$V_{\phi}(R) = \sup_{\sigma(R)} \sum_{r \in \sigma(R)} v_{\phi}(r).$$

The number $V_{\phi}(R)$ is called the *variation* of ϕ on R , and

$$\phi \text{ is BV means } V_{\phi}(X^1) < \infty.$$

As for the concept of AC, to do minor violence to the customary definition:

ϕ is AC means: (a) for every $\epsilon > 0$ there is a $\delta(\epsilon)$ such that if r_1, \dots, r_n is a set of closed intervals with interiors disjoint in pairs and $\sum |r_i| < \delta(\epsilon)$, then $\sum V_{\phi}(r_i) < \epsilon$.

$$(b) \quad V_{\phi}(R) = \sum_{r \in \sigma(R)} V_{\phi}(r), \text{ for any } \sigma(R).$$

It is well known, however, that (b) is always true, hence it is redundant in this definition and is here displayed only for future use.

9. Final classical theorem for curves and future criterion for surfaces. Now that these definitions have been recorded it is important to observe that in the case of a mapping $f: X^1 \rightarrow E^3$, if σ is a subdivision of X^1 and

$$g_{\sigma}(f) = \sum_{r \in \sigma} [v_r^1(r) + v_r^2(r) + v_r^3(r)]^{1/2},$$

then $g_{\sigma}(f)$ is the length of the *inscribed polygon* determined by σ . Moreover, if

$$G(f) = \sup_{\sigma} g_{\sigma}(f)$$

then one has the final classical result:

$$C_4: L^1(f) = G(f).$$

(Notice that C_4 implies $L^1(\phi) = G(\phi) = V_{\phi}(X^1)$.)

10. Surfaces. *In now considering the situation for surfaces, the same general outline is employed except that f is now a mapping from the unit square X^2 into Euclidean 3-space E^3 , and in contradistinction to the earlier convention, it is now assumed that Y^1 , Y^2 , and Y^3 are coordinate planes in E^3 . If $p^i: E^3 \rightarrow Y^i$ is the perpendicular projection, then let $f^i = p^i f$ as before. The mapping $f^i: X^2 \rightarrow Y^i$ is a so-called flat transformation from the unit square into a plane, $i=1, 2, 3$.*

The classical integral formula (see §3) remains unchanged except that J^i is now the Jacobean of the flat transformation f^i and, of course, 2-dimensional integration is understood. The formula will be designated by $I^2(f)$ and the earlier conventions concerning its use apply. (See §3.)

It has been stated that outstanding problems in Lebesgue area arise in the search for definitions of BV and AC for flat transformations which will provide results analogous to the classical theorems C_1 , C_2 , C_3 , and C_4 (see §§5 and 9). *These theorems thus become criteria for the suitability of proposed concepts of BV and AC.*

Initial attacks on the problem were made by de Geöcze, and though he was far from being entirely successful in his venture, it came to be considered a safe judgment—even in a field where history has shown the most plausible conjectures to be fraught with disaster—that modifications of his ideas would stand the test of these criteria. In a sense, appropriate concepts have long been available—the difficulty has been in providing proofs.

In defining BV and AC for flat transformations, the development will be by analogy from the 1-dimensional case and it must be stated that considerable expository license will be taken with the ideas of Cesari [3, 4, 5, 6, 7], Radó [18, 19, 20, 24, 25, 26, 27], and Reichelderfer [27, 28].

In accord with the modifications introduced above, the mapping ϕ (cf. §6) is now considered to be a flat transformation from the unit square X^2 into a plane E^2 . (It is therefore of the same type as the mappings f^i found above.) The interval function of the 1-dimensional case is to be replaced by a function of the closure r , of a Jordan region bounded by a simple closed polygon $\dot{r} \subset X^2$.

In the case for curves $v_\phi(r)$ was interpreted as the 1-dimensional measure of a certain set in $E^1 - \phi(\dot{r})$ (see §7). By direct analogy, $v_\phi(r)$ is now defined to be the 2-dimensional measure of the corresponding set $[E^2 - \phi(\dot{r})] \cap \bigcap_{\Gamma} \gamma(r)$ where Γ is the family of mappings $\gamma: r \rightarrow E^2$ which agree with ϕ on \dot{r} .

The variation $V_\phi(R)$ of ϕ on R is defined in exactly the same manner (see §8) and there is no change in the definitions of BV, AC, and $G(f)$.

(See §§8 and 9.) In view of this fact these definitions are not recorded anew. It is to be noted, however, that item (b) in the definition of AC is *not* redundant. In other words, what was really a theorem in the 1-dimensional case now becomes an essential portion of the definition of AC. (See Cesari [6].)

It should be noted that the functional $g_\sigma(f)$ can no longer be interpreted as the area of an inscribed polyhedron even if σ is a triangulation of X^2 . The functional $G(f)$, however, may be interpreted as one of the possible definitions for the *de Geöcze area* of the surface represented by f .

11. Lebesgue area. Prior to considering how these concepts fit into the framework of the criteria C_1 , C_2 , C_3 , and C_4 (see §§5 and 9), it is proper to define $L^2(f)$, the Lebesgue area of the surface represented by f . A mapping $\mathfrak{p}: X \rightarrow E^3$ is said to be *polyhedral* (=quasi linear, see McShane [16]) if there is a triangulation T of X with the property that $\mathfrak{p}^i = \mathfrak{p}^i \mathfrak{p}$ is linear on each triangle of T . (See §10.) The elementary area $E(\mathfrak{p})$ of the polyhedron given by \mathfrak{p} is simply the sum of the areas of the triangles $\mathfrak{p}(\Delta)$, $\Delta \in T$, and

$$L^2(f) = \inf \left[\liminf_{n \rightarrow \infty} E(\mathfrak{p}_n) \right],$$

where the infimum is taken for all sequences \mathfrak{p}_n converging uniformly to f . It is not difficult to see that if f and g represent the same surface (see §4) then $L^2(f) = L^2(g)$. (There is an obvious lack of analogy here between the definitions of length (see §4) and area. In point of fact, length could equally well have been defined by 1-dimensional considerations precisely analogous to the 2-dimensional considerations above. This was not done since the definition of §4 is better known. It is not to be supposed, conversely, that area can be defined by considerations precisely analogous to those employed in the definition of length given in §4. For a discussion of this matter see Youngs [31].)

12. Generalized Jacobians. In addition to the definition of Lebesgue area, it is necessary to say a word about the classical integral formula $I^2(f)$, where the symbols J^i now stand for the Jacobian of f^i , $i = 1, 2, 3$. (See §§3 and 10.) For a variety of reasons it is necessary to generalize the concept of Jacobian.

It will be recalled that the *absolute* value of the ordinary Jacobian of a flat transformation $\phi: X^2 \rightarrow E^2$ measures the local "ratio of magnification" accorded by the mapping. By analogy from the 1-dimensional case (see end of §9) one can hope that $V_\phi(r) = L^2(\phi|_r)$ —where

$\phi|_r$ is the mapping ϕ considered only on r . Hence a natural generalization is to define

$$\mathcal{J}(x) = \lim_{|s| \rightarrow 0} \frac{V_\phi(s)}{|s|}$$

where s is a square with sides parallel to the axes of X^2 and $x \in s^0$. (If the ordinary Jacobian $J(x)$ exists almost everywhere then the same is true of $\mathcal{J}(x)$, and $|J(x)| = \mathcal{J}(x)$ almost everywhere; see Cesari [4] and Radó [26].)

If the ordinary Jacobians in the classical integral formula $I^2(f)$ are replaced by the generalized Jacobians, then the new formula is designated by $\mathfrak{J}^2(f)$ and the earlier conventions concerning its use apply (see §§3 and 10).

13. Results for surfaces. As to the four criteria (see §§5 and 9) it is now possible to state that these statements all hold if I^1 is replaced by \mathfrak{J}^2 and L^1 by L^2 .

This simple assertion is by no means simple to prove. It represents the concerted efforts of many mathematicians over approximately half a century and, in point of fact, C_3 has only recently been announced by Cesari [8].

In proving these theorems one must naturally work with the basic function $v_\phi(r)$, and it must be stated that an equivalent definition in terms of the *degree of a mapping* is employed. (See, for example, Cesari [3] and Radó [26].) This equivalent definition need not concern us. The point is that in consequence of this fact, it is only to be expected that standard theorems on the degree will be encountered. What is perhaps unexpected is that further topological methods, to be known in this context as *topological adventures*, must be brought to bear on the situation. In any event it is the topological adventures *alone* which will be the center of attention.

It should also be stated that in view of technical complexities involved in the proofs and, indeed, the sheer lengths of the proofs themselves, no details can be undertaken, and the *application* of the topological adventures to be considered can at best be implied.

14. Inequality of Cesari and criterion C_1 . Of the four theorems C_1 perhaps deserves special attention in view of its difficulty and the bearing its solution has on some of the others, notably C_4 .

That $L^2(f) < \infty$ implies f^i is BV was shown independently by Cesari [3] and Radó [24] during the war years. Cesari, however, proved both parts of C_1 using an argument which culminated in the *basic inequality*,

$$V_{f^i}(X^2) \leq L^2(f) \leq V_{f^1}(X^2) + V_{f^2}(X^2) + V_{f^3}(X^2), \quad i = 1, 2, 3.$$

From this inequality the issue is settled at a glance. This superb result must certainly rank as an accomplishment of the first magnitude.

The right half of the Cesari inequality is much more difficult to prove than the left. On the other hand, it will follow immediately from the definition of $L^2(f)$ if it can be shown that for each integer n there is a polyhedral mapping $p_n: X \rightarrow E^3$ such that

$$\begin{aligned} \text{(a)} \quad & d(f(x), p_n(x)) < 1/n, & x \in X. \\ \text{(b)} \quad & E(p_n) \leq V_{f^1}(X^2) + V_{f^2}(X^2) + V_{f^3}(X^2). & \text{(See §11.)} \end{aligned}$$

To obtain (a) by itself is a triviality, but to obtain it in conjunction with (b) is no easy task and involves carefully fitting together a polygonal mapping by a highly delicate argument. A salient step deeply imbedded in the chain of reasoning is concerned with a property of Euclidean 3-space which should be of some interest.

In E^3 let M consist of the coordinate axes X, Y, Z . With $\delta > 0$, define M_δ to be the union of the four lines:

$$\begin{aligned} X_\delta: y = 0, z = -\delta. & \quad Y_\delta: x = 0, z = \delta. \\ Z_\delta^+: x = \delta, y = \delta. & \quad Z_\delta^-: x = -\delta, y = -\delta. \end{aligned}$$

Now suppose that one is given a closed path in $E^3 - M$ at a distance greater than δ from M . A lemma of Cesari [2] states that *if the path is contractible in $E^3 - M_\delta$ then it is also contractible in $E^3 - M$ and conversely*. (This form of the statement is due to Eilenberg [10] who has also considerably shortened the original proof.)

The inequality of Cesari shows, by the way, that

$$\text{(a)} \quad L^2(f^i) = V_{f^i}(X^2), \quad i = 1, 2, 3,$$

and hence

$$\text{(b)} \quad L^2(f^i) \leq L^2(f) \leq L^2(f^1) + L^2(f^2) + L^2(f^3), \quad i = 1, 2, 3.$$

But (a) can be proved independently, and hence the Cesari inequality follows if the same is true of (b).

The first half of (b) is obvious and the second has the deceptive air of appearing to be simpler to prove than the Cesari inequality since the functional L^2 alone appears. However, it is not known whether (b) can be proved directly or not.

15. Middle space topology. It is not to be supposed however that C_1 was proved without further topological adventures. For example a close scrutiny of the character and disposition of the inverse sets

$f^{-1}(y)$, $y \in E^3$, was required. Since studies of this character also appear in connection with the other theorems, and here analytic topology plays a major role, the matter cannot be dismissed with cursory mention. (For reasons which will become apparent this whole branch of the theory has aptly been called *middle space topology* by Radó [23].)

The basic concept involved is that of a *monotone-light factorization* of a mapping. If $f: X \rightarrow Y$ is a mapping from a compact space into a metric space, then the Eilenberg-Whyburn Factor Theorem (see Whyburn [30]) states that there is a pair of mappings $m: X \rightarrow \mathfrak{X}$ and $l: \mathfrak{X} \rightarrow Y$ such that:

- (a) $f = lm$.
- (b) m is monotone ($= m^{-1}(x)$ is a continuum, $x \in X$), and $m(X) = \mathfrak{X}$.
- (c) l is light ($=$ no component of $l^{-1}(y)$ consists of more than one point, $y \in Y$).

The space \mathfrak{X} is called the *middle space* of the *monotone-light factorization* lm of f .

In the study of Lebesgue area it became evident that the structure of the middle space \mathfrak{X} —the image of the monotone factor—was of considerable importance in contradistinction to the apparently paradoxical fact that the structure of $f(X)$ —the image of f —had relatively little to do with the matter.

For example, it was observed by Morrey [17] that *if X is a 2-cell and $L^2(f) < \infty$ then it is possible to find a new representation $g: X \rightarrow E^3$ such that the Lebesgue area is given by the classical integral formula $I^2(g)$.*

16. Representation problem. This is perhaps as good a time as any to state that the general problem of finding new representations for a surface had earlier attracted the attention of Kerékjártó [14]. He considered the case in which X is a 2-sphere while $f_1: X \rightarrow E^3$ and $f_2: X \rightarrow E^3$ are representations of the same surface, that is to say, are Fréchet equivalent (see §4). Under these conditions, Kerékjártó showed that *there is a monotone-light factorization lm_i of f_i with common middle space \mathfrak{X} , $i = 1, 2$.* (Note that the factorizations also have a common light factor.)

It was initially supposed that the converse of this theorem is true. This is *not* the case (see Youngs [32]), and it should be mentioned that it is the converse which is important in these problems. However, Kerékjártó's work initiated the search for conditions which are both necessary and sufficient for $f_1 \sim f_2$. (See §4.) This is the *representation problem* for surfaces. The solution requires the use of algebraic topology—in this presentation the Čech cohomology theory with the additive group of integers as the coefficient group will be

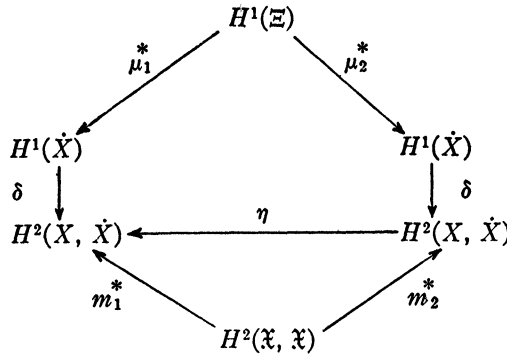
employed. The result in question is the following and is initially stated for the case X is a 2-cell with boundary \dot{X} .

$f_1 \sim f_2$ if and only if:

1. There is a monotone-light factorization lm_i of f_i with common middle space \mathfrak{X} , $i=1, 2$.

2. If $m_i = m_i|_{\dot{X}}$ then there is a monotone-light factorization $\lambda\mu_i$ of m_i with common middle space \mathfrak{E} , $i=1, 2$. (This implies $m_1(\dot{X}) = m_2(\dot{X}) \equiv \mathfrak{X}$.)

3. If $m_i^*: H^2(\mathfrak{X}, \mathfrak{X}) \rightarrow H^2(X, \dot{X})$ and $\mu_i^*: H^1(\mathfrak{E}) \rightarrow H^1(\dot{X})$ are the homomorphisms induced by the mappings m_i and μ_i respectively, $i=1, 2$, then there is an automorphism $\eta: H^2(X, \dot{X}) \approx H^2(X, \dot{X})$ such that with respect to the following diagram



one has $\delta\mu_1^* = \eta\delta\mu_2^*$ and $m_1^* = \eta m_2^*$.

In the event X is a 2-sphere, $\dot{X} = 0$, and the statement of the result is simplified in the obvious manner (Youngs [32]).

17. Additivity theorem. The first active use made of the solution of the representation problem was in the proof of an *additivity theorem for the Lebesgue area*. Suppose $f: X \rightarrow E^3$ represents a surface with X a 2-sphere, and lm is a monotone light factorization of f with middle space \mathfrak{X} . (See §15.) The collection of true cyclic elements of \mathfrak{X} is at most denumerable. (See Whyburn [30].) Let them be represented by $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \dots$. It is well known that there is a unique monotone retraction $r_n: \mathfrak{X} \rightarrow \mathfrak{C}_n$, $n=1, 2, 3, \dots$.

If $f_n = lr_n m$, $n=1, 2, 3, \dots$, then this class of mappings is said to be the *cyclic decomposition* of f , and the *cyclic additivity theorem* for Lebesgue area states that

$$L^2(f) = \sum L^2(f_n),$$

where the symbols on the right are interpreted to mean 0 if there are no true cyclic elements.

It should be stated that the theorem is also true in the event X is a 2-cell as has been shown by Radó [26] and Hiesel [13].

The cyclic additivity theorem shows that $L^2(f) = 0$ if the middle space has no true cyclic elements, that is to say, if \mathfrak{X} is a dendrite. Radó [22] has also proved the converse of this theorem thus providing an elegant characterization for surfaces of zero area.

18. Criterion C_4 . It can be shown that the Geöcze area is also cyclicly additive, and hence the equality of the two definitions of area, that is, a solution of C_4 (see §9) follows if $G(f_n) = L^2(f_n)$, $n = 1, 2, 3, \dots$. This has been done independently by Cesari [5, 7] and Radó [24, 26] in the event $L(f_n) < \infty$ —it is topologically interesting to note that the proof depends heavily on the fact that the surface represented by f_n can also be represented by a mapping g_n which is light on the interior of X^2 , $n = 1, 2, 3, \dots$.

In the event $L^2(f_n) = \infty$ Cesari completed the solution of this celebrated problem by employing his inequality (see §14) together with the obvious fact that

$$V_{f_n^i}^i(X^2) \leq G(f_n), \quad i = 1, 2, 3.$$

For now

$$L(f_n) \leq V_{f_n^1}^1(X^2) + V_{f_n^2}^2(X^2) + V_{f_n^3}^3(X^2) \leq 3G(f_n).$$

Consequently $L^2(f_n) = \infty$ implies $G(f_n) = \infty$.

19. Criteria C_2 and C_3 . The lack of recent specific mention of C_2 and C_3 (see §5) may lead one to suppose that here, at any rate, no topological adventures are required. While, relative to the other theorems, there is a certain element of truth to this surmise, it is not to be supposed that C_2 or C_3 can be proved in the body of analysis. For example, a basic step in the proof of C_2 is the fact that C_4 is true in particular if $L^2(f) < \infty$. (See Cesari [4] and Radó [26].) Hence indirectly at least the most powerful methods of middle space topology are required. As for C_3 , Cesari's announcement shows that he proves this by first generalizing the result of Morrey already stated in part. (See §15 and Cesari [8, 9].) The argument again may be based upon middle space topology. Hence in a certain sense the most that can be said is that no topological adventures are required which are spectacularly different from those already employed in the proofs of C_1 and C_4 .

Relative to C_3 , a portion of the results of Cesari shows that if $L^2(f) < \infty$ then there is a closed set $H \subset X$ and a representation g of the surface such that $L^2(g) = \int_H [(J_1^1)^2 + (J_2^2)^2 + (J_3^3)^2]^{1/2} dx$. (See §3 and

Cesari [8].) The subscript s is here employed to indicate that the ordinary Jacobians are taken relative to the representation g but that the concept of differentiation is appropriately generalized using the work of Stepanoff.

To date this result is the closest approach to an unsolved problem of obvious interest. *If $L^2(f) < \infty$ is there a representation $g \sim f$ (see §4) such that $L^2(g) = I^2(g)$?* In other words is every surface of finite Lebesgue area so representable that the ordinary classical integral formula applied to this representation yields the Lebesgue area? Radó [21] has shown that if $I^2(f)$ has meaning, then $L^2(f) \geq I^2(f)$, and it is known that every surface, regardless of its Lebesgue area, has a representation g for which $I^2(g) = 0$. (See Youngs [33].) If the above question is answered in the affirmative it will certainly provide a beautiful link between the classical ideas of the last century and the modern theory of Lebesgue area. (Added in proof: Cesari has recently shown that such representations exist.)

20. Generalizations. The statement of this unsolved problem concludes this survey of some of the topological adventures forced by these classical problems. However, a few concluding remarks will be made on generalizations in two directions:

1. The problem of mappings from cells of dimension higher than 2.
2. The problem of mappings from compact connected 2-manifolds.

The first problem has attracted the attention of Federer who has reported (in correspondence) a fruitful generalization of the concept of BV for mappings from an n -cell into Euclidean n -space. His concept is, in point of fact, a modification of the method by which Radó defines BV in the 2-dimensional case; namely by using the concept of essential multiplicity which he brought to bear on these problems twenty years ago. (See Radó [18].)

Without going into details, Federer's ideas may be sketched as follows:

Suppose $\phi: X^n \rightarrow E^n$ is a mapping from an n -cell into Euclidean n -space. For $y \in E^n$ let $S_\epsilon(y)$ be the open sphere of radius ϵ about y , and U_1, U_2, U_3, \dots be those components of $\phi^{-1}(S_\epsilon(y)) \equiv W$ having the property that $U_i \cap \dot{X}^n = 0$, $i = 1, 2, 3, \dots$ (\dot{X}^n is the boundary of X^n).

If $=\phi|_W$ then $\bar{\phi}: W \rightarrow S_\epsilon(y)$ and induces a homomorphism $\bar{\phi}^*: H^n(S_\epsilon(y)) \rightarrow H^n(W)$ between the Lefschetz groups. (See Cartan [1].) But $H^n(W)$ is naturally isomorphic to the weak direct product $PwH^n(U_i)$. If this isomorphism is denoted by η , and generators s and u_i are selected for the infinite cyclic groups $H^n(S_\epsilon(y))$ and $H^n(U_i)$ respectively, $i = 1, 2, 3, \dots$, then

$$\bar{\phi}_\epsilon^*(s) = (k_1 u_1, k_2 u_2, k_3 u_3, \dots)$$

where k_i is an integer and different from zero for only a *finite* number of subscripts $\kappa_\epsilon(y)$.

Now $\kappa_\epsilon(y)$ is independent of the selection of generators and does not decrease as $\epsilon \rightarrow 0$. If $\kappa(y) = \lim \kappa_\epsilon(y)$, then in case $n=2$, Radó defines it to be the *essential multiplicity* of y under ϕ . He further defines $V_\phi(X^2) = \int_{E^2} \kappa(y) dy$ and shows that

$$L^2(\phi) = \int_{E^2} \kappa(y) dy.$$

If $n > 2$ then $L^n(\phi) \geq \int_{E^n} \kappa(y) dy$ and the equality need not hold even if $n=3$. ($L^n(\phi)$ is the n -dimensional Lebesgue "area" of the "surface" represented by ϕ and is defined by obvious modifications of §11.)

On the other hand, Federer defines $\lambda_\epsilon(y) = \sum |k_i|$, and $\lambda(y) = \lim \lambda_\epsilon(y)$. He then proves the fundamental equality:

$$L^n(\phi) = \int_{E^n} \lambda(y) dy.$$

The publication of this beautiful result will certainly be awaited with interest.

It should be mentioned, under this first direction of generalization, that results for mappings—even from 3-cells—comparable in scope and importance with those results in the 2-dimensional case which are concerned with middle space topology lead immediately to basic unsolved problems; for example, the characterization of a 3-cell. One is perhaps permitted, therefore, to take a dim view of the possibility of startling developments along the lines of middle space topology in the immediate future.

The second direction along which generalizations may be made, especially insofar as the corresponding representation problem is concerned (see §16), has been of personal interest during the past few years. This general representation problem has been solved and uses the fundamental work done by Roberts and Steenrod [29] in characterizing the monotone image of a 2-manifold.

This is hardly the place for a discussion of the solution, but it may be of interest to note that a theorem corresponding to the cyclic additivity theorem goes through, and yields a generalization of the theorem of Radó concerning surfaces of zero area. (See §17.) Specifically, *if $f: X \rightarrow E^3$ is a mapping from a compact connected 2-manifold and lm is a monotone-light factorization of f with middle space \mathfrak{K} (see*

§15), then $L^2(f) = 0$ if and only if $H^2(\mathfrak{X}) = 0$; or alternately, $L^2(f) = 0$ if and only if $\dim \mathfrak{X} < 2$. A certain "slenderness" of the middle space is thus a necessary and sufficient condition for the Lebesgue area to be zero.

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