that "the problem of the infinitesimal stability of the periodic solutions of nonlinear systems always leads to a Hill equation." Following this thought, the author reduces the problem of stability to the analysis of the local behavior of the variational equations (of the Hill type) in the various parts of the response curve and applies this procedure to the Duffing equation. It turns out, however, that, on this basis, the free oscillations are unstable. Having ascertained this seemingly paradoxical result, the author ascribes it to the fact that the criterion of the "infinitesimal stability" (that is, the stability in the sense of the variational equations) ought to be replaced by that of the orbital stability. In fact, after a somewhat delicate argument, in §7, the author proves this point. In spite of this, the reader, particularly the beginner, must inevitably feel somewhat confused as to when to use one criterion and when to use the other. This question does not seem to find a definite answer in the text, probably because the author, as he said in his introduction, had to curtail considerably the theory of stability owing to lack of space. It seems, however, sufficiently simple to show that if the differential equations are referred to the "amplitude-phase" plane (a, ϕ) instead of the usual $(x, \dot{x})$ phase plane (namely, $da/dt=f_1(a, \phi)$, $d\phi/dt=f_2(a, \phi)$) the singular point $f_1(a_0, \phi_0)=f_2(a_0, \phi_0)=0$ in this case represents the stationary periodic motion (if $a_0\neq 0$) and the variational equations ("the infinitesimal stability") give precisely the orbital stability in such a case, without any necessity of applying the theory of characteristic exponents of Poincaré. Reduction to this form is always possible if the differential equations do not contain time explicitly.

N. MINORSKY

Differential algebra. By Joseph Fels Ritt. (American Mathematical Society Colloquium Publications, vol. 33.) New York, American Mathematical Society, 1950. 8+181 pp. $4.40.

It was a gigantic task that J. F. Ritt undertook twenty years ago: to give the classical theory of nonlinear differential equations a rigorous algebraic foundation. Emmy Noether and her school had done the same thing for the theory of algebraic equations and algebraic varieties, but differential equations are much more difficult than algebraic equations. Luckily, Ritt has gathered around himself a whole school of able collaborators: Raudenbusch, Strodt, Kolchin, Howard Levi, Gourin, R. M. Cohn.

The present book is not just a revised and enlarged edition of the author's *Differential equations from the algebraic standpoint* (Colloquium Publications, vol. 14). It is written from a much higher point of view and based upon new principles. The first edition was a tentative, this a classic.

One important new feature is that the whole theory is now based upon the Ritt-Raudenbusch *basis theorem.*

Let $\mathcal{J}$ be a field in which an operation of differentiation is performable, for example a field of functions of one variable x closed with respect to differentiation. Let y_i $(i=1, \cdots, n)$ be symbols of unknown functions, and y_{ij} symbols of their derivatives (for example $y_{i0}=y_i$ and $y_{i1}=y_i'$). Polynomials in the indeterminates y_{ij} are called d.p. (differential polynomials). They form a ring in which a differentiation is defined.

Let Σ be any infinite set of d.p. A finite subset Λ is said to form a *basis* of Σ if for every d.p. A in Σ, a power A^p is a sum of multiples of elements of Λ and their derivatives. The basis theorem now states: *Every set* Σ *has a finite basis.*

The basis theorem was implicit in Ritt's work, but it was Raudenbush who brought it to the present complete form. By placing this theorem right at the beginning, in Chapter I, the great line of thought is made much clearer than it was in the original treatment.

Starting from the basis theorem, the theory of differential ideals is developed up to the representation of "perfect ideals" as intersections of prime ideals.

If for the indeterminates $y_1, \cdots, y_n$ field elements $\eta_1, \cdots, \eta_n$ are substituted, it may happen that all d.p. of a set Σ become zero. In this case, the set $\{\eta_1, \cdots, \eta_n\}$ is called a *zero* or *solution* of Σ. The totality of all zeros is the *manifold* of Σ.

Chapter II deals with manifolds and their decomposition into irreducible manifolds. Just as in algebraic geometry, the *dimension* of an irreducible manifold can be defined as the maximum number of parametric indeterminates among the y_i.

For one single differential equation $F=0$ a distinction between *singular* and *nonsingular* solutions is introduced. In the manifold of F, there is always one irreducible component, called the *general solution* which does not consist of singular solutions only. Perhaps *nonsingular component* would be a better expression.

The discussion of the solutions of one single differential equation is continued in Chapter III. The "low power theorem" gives the necessary and sufficient condition that the nonsingular component of one differential equation be at the same time a component of another differential equation. This theorem solves a problem first treated by Laplace and Poisson by heuristic methods.

In Chapter IV, systems of algebraic equations are treated by a new algorithmic method. The results of this chapter are used in Chapter V to give an algorithmic treatment of various questions connected with finite systems of differential equations. For the case of a field $\mathcal{J}$ consisting of analytic functions, a very useful approximation theorem is proved.

Chapter V deals with constructive methods and tests.

In Chapter VI, the case of a field of analytic functions is treated by analytic methods. For this case, another proof of the low power theorem is given.

Chapter VII deals with intersections of algebraic differential manifolds, especially with their dimensions. A result of Jacobi proves true in some special cases, but false in general.

Chapters VIII and IX deal with partial differential equations. In Chapter VIII, a very important existence theorem, due to Riquier, is proved. In Chapter IX this theorem is used to extend some of the main results of the preceding chapters to partial differential polynomials.

B. L. VAN DER WAERDEN

Transcendental numbers. By Carl Ludwig Siegel. (Annals of Mathematics Studies, no. 16.) Princeton University Press, 1949. 8+102 pp. $2.00.

As the author states in a short preface, this book is based on lectures given at Princeton in 1946. In Chapter I, *The exponential function*, proofs are given of the irrationality of e and π, and then a general method is introduced.

Let $\rho_1, \cdots, \rho_m$ be complex numbers, $n_1, \cdots, n_m$, non-negative integers, and let

$$N + 1 = \sum_{k=1}^{m} (n_k + 1).$$

It is shown that polynomials $P_1(x), \cdots, P_m(x)$ of degrees $n_1, \cdots, n_m$, respectively, may be determined uniquely (up to a constant factor) such that the function

$$R(x) = \sum_{k=1}^{m} P_k(x) e^{\rho_k x}$$

vanishes at $x = 0$ of order N. Such a function is called an approximation form. An explicit formula for $R(x)$ as a multiple integral provides an upper bound for $|R(1)|$ and shows that $R(1) > 0$ when $\rho_1, \cdots, \rho_m$ are real.