BOOK REVIEWS

Contributions to the theory of games. Ed. by H. W. Kuhn and A. W. Tucker. (Annals of Mathematics Studies, no. 24.) Princeton University Press, 1950. 16+201 pp.

The modern theory of games had its inception in von Neumann's paper in the Mathematische Annalen for 1928, and was developed and popularized in the book by von Neumann and Morgenstern. Its great popularity is due to its mathematical interest and to two other reasons. One of these is the connection between the theory of games and the theory of statistical decision functions initiated independently by Wald in 1939. The other is the unhappy state of the world in these last years, and the abundance of wars, hot and cold. To the mathematician confronted with a problem in strategy it is very appealing to have recourse to a theory and an established procedure like seeking a minimax solution. Much of what is now called (military) operations analysis seems to be simply an application of the theory of games.

The present study comprises a number of diverse papers on the theory of games, with a comprehensive introduction by the editors. In this introduction the editors give a brief description of each paper and conclude with a list of questions for which, in their opinion, it is very important to have answers forthcoming. Both the summary and list of questions are very clearly and adequately put, and can serve the busy reader as a guide to further reading in this volume and elsewhere. The excellent introduction and the general theme furnish the bonds of unity for the papers in the volume, many of which were originally submitted to the Annals of Mathematics and transferred here by common consent. Since the individual papers will be reviewed in Mathematical Reviews, it seems best to this reviewer to content himself here with a rather general description of the contents of the volume.

The first paper, by Weyl, is a reprint of his 1935 paper on the equivalence of the two definitions of the convex closure of a finite set of points in Euclidean space. The second paper, also by Weyl, gives a proof of the Main Theorem (of von Neumann) when all elements of the payoff matrix lie in an ordered field. (The Main Theorem is the one which asserts that a finite game is determined when mixed strategies are admitted.)

Continuing Part I, on finite games, Shapley and Snow give a characterization of the optimal (mixed) strategies in terms of submatrices of the payoff matrix. In principle this characterization gives an algorithm for computing the optimal solutions, but in practice the computational effort required will be too great. Brown and von Neumann give a finite system of first order differential equations in one independent variable t, with the property that any limit point, as $t \rightarrow \infty$, of a solution, is an optimal strategy. At least one limit point exists and it is said that the procedure of solving the differential equations can be mechanized with relative ease.

The procedure of Brown and von Neumann makes use of the fact that the solution of any finite game can be made to depend upon the solution of a symmetric game, that is, one with a skew-symmetric payoff matrix. In another paper (no. 7) Gale, Kuhn, and Tucker give a superior method of "symmetrizing" a game, and a characterization of the optimal solutions of a symmetric game.

Bohnenblust, Karlin, and Shapley in one paper, and Gale and Sherman in another, give geometric characterizations of the totality of solutions of a zero-sum two-person game. There is some overlap between the two papers.

Kuhn (paper no. 9) and Nash and Shapley (paper no. 10) analyze simplified versions of poker.

In paper no. 8 Gale, Kuhn, and Tucker study what happens to games under certain classes of transformations of their matrices. Such transformations are of great importance in actual computational problems.

J. C. C. McKinsey discusses the question of strategic equivalence of n-person ($n > 2$) games and proves some results on this subject. Roughly speaking, two games are strategically equivalent if the bargaining powers of the players and their tendencies to form coalitions are the same.

A paper (no. 12) by Karlin initiates the second part of four papers on infinite games. The principal result seems to be a slightly stronger version of Theorem 2.23 of Wald's book *Statistical decision functions*. (The two authors did their work independently and approximately at the same time.) The reviewer is informed that the second part of the paper contains the result that if one admits strategies which are only finitely additive then every zero-sum two-person game is determined. The reviewer cannot vouch for this, however, as he gave up the task of reading this paper before the end. This paper, which seems to contain interesting results, is written in a singularly inept and chaotic manner.

Paper no. 13 by Bohnenblust and Karlin generalizes to Banach spaces theorems of Ville and Kakutani which were useful tools in proving the Main Theorem for finite games.

The volume concludes with two papers: Paper no. 14, by Dresher, Karlin, and Shapley, is concerned principally with zero-sum two-person games with the kernel $K(x, y) = \sum_{i,j} a_{ij}r_i(x)s_j(y)$, where the functions $r_i(x)$ and $s_j(y)$ are continuous, and x and y, the pure strategies of the two players, are points on the unit interval. Paper no. 15 by Bohnenblust, Karlin, and Shapley, discusses zero-sum two-person games with payoff function $M(x, y)$, which is such that, for every x in an arbitrary set A, $M(x, y)$ is a continuous convex function of y. The y-player is shown to have an optimal pure strategy (y is a point in a compact, convex n-dimensional region of a finite Euclidean space). The main result of the paper is that the x-player has an optimal mixed strategy which assigns probability one to a set which contains at most $(n+1)$ points x.

This volume has the merit of bringing together between two covers a large number of interesting papers on game theory which would otherwise be inconveniently scattered over many periodical issues. Such enterprises could profitably become a feature also in other branches of mathematics. The editors deserve much credit for a painstaking job, and for their lucid and stimulating introduction.

J. Wolfowitz

Hydrodynamics, a study in logic, fact, and similitude. By Garrett Birkhoff. Princeton University Press, 1950. 14+186 pp. $3.50.

The material of this book formed the contents of a series of lectures at the University of Cincinnati in 1947, and this perhaps accounts for the unusual and unconventional choice of material for a book with the title *Hydrodynamics*. There are five chapters in the book, most of which have only a loose connection with the others.

The reviewer found it difficult to understand for what class of readers the first chapter was written. For readers who are acquainted with hydrodynamics the majority of the cases cited as paradoxes belong either in the category of mistakes long since rectified, or in the category of discrepancies between theory and experiment the reasons for which are also well understood. On the other hand, the uninitiated would be very likely to get wrong ideas about some of the important and useful achievements in hydrodynamics from reading this chapter. In the case of air foil theory, for example, the author treats only the negative aspects of the theory. It has always seemed to the reviewer that the Kutta-Joukowsky theory of airfoils is one of the most beautiful and striking accomplishments in applied mathematics. The fact that the introduction of a sharp trailing edge makes possible a physical argument, based on consideration of the effect of viscosity, that